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A Rigourous Mathematical Framework for Computing a Sustainability Ratio: the Emergy

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ABSTRACT. The computational problem of emergy within a general system of interconnected processes at steady state is a subject of interest in literature today. When there is no co-product the proposed method coincides with the Track Summing Method of Tennenbaum which was developed precisely for interconnected networks with feedbacks and splits of emergy. As the underlying algebraic structure of the Tennenbaum's method is the linear algebra, it is not well-suited to account for the co-product problem which induces the idempotent operator max. Thus, authors have chosen another underlying algebraic structure which is the idempotent semiring structure (i.e. a semiring equipped with an idempotent addition). This method is divided into two parts. The first part is the emergy flow enumeration, where paths from an emergy source to the input of a given process are enumerated avoiding double counting of emergy assignation. This part is a path-finding problem which is a slight modification of gerbier of null square approach to find elementary/simple paths in a graph. The second part evaluates the emergy flowing between two components of the system. It is a quantitative part in which the problem of avoiding double counting split and co-product flows are dealt with by introducing a way to mark splits and co-products flows. The method is partially parallelizable. However, the method enumerates paths on a graph thus, in worst cases, its complexity is not polynomial. This paper provides a rigorous framework based on an axiomatic basis to conduct the emergy evaluation of an emergy graph.

Keywords: sustainability, emergy algebra, graph, formal language, max-plus algebra

1. Introduction

The first principle of thermodynamics states that heat and power are two forms of the same quantity: energy. Carnot (1796 ~ 1832) wrote that: "A fire machine will be able to produce a positive quantity of mechanical work only at the price of a fall of temperature". This idea is expressed in substance by: "There is a price to pay to Nature to produce work", introducing the quality of energy. Generally speaking, exergy characterizes the quality of energy. Energy and exergy are two thermodynamic state functions of thermodynamic state variables (for example, pressure, temperature, etc.) and consequently do not depend on the process. Hence, oil, natural gas and coal are featured by their physical properties, such as low heating value. Nevertheless, one major dimension is forgotten in this approach, those fuels are time-derivatives of wood (trees) decomposition (burning during thousand of millenaries near the earth magma), see two recent papers Brown and Ulgiati (2010) and Brown et al. (2011). As a result, apart from nuclear energy and geothermic, all major energy sources can be, directly or indirectly, considered as a "solar energy tank".

According to Odum (1996) emergy, spelled with an "m", can be defined as the total solar equivalent energy/exergy of one form that was used up directly or indirectly in the work of making a product or a service. The physical dimensions are those of the energy. So emergy provides a general framework to account for both ecological and human activities which make it an attractive concept. However, one of the main drawbacks of emergy analysis is the difficulty to obtain a clear procedure for computing emergy. The main problem is to avoid double counting within a system of interconnected processes with feedbacks, splits and co-products. When working with emergy, the comparison of interconnected processes/components can be based on the same fundamentals and provides reliable sustainability development dimensionless numbers. The idea of emergy is based on the maximum power principle originally stated by the biologist Lotka (1922). On the one hand, emergy is not a thermodynamic state function and consequently depends on its pathway or its history. Emergy assessment does not "add" two energies of different qualities but add the solar embodied energy of a source. Hence, the main interest (advantage) of emergy is the comparison of two processes/products with multi-sources (renewable and/or non-renewable sources), see Brown and Ulgiati (1997). On the other hand, system boundaries must be clearly detailed.

As mentioned in Hau and Bakshi (2004) even if the idea of emergy is attractive, only Odum and a small circle of coworkers have developed the notion of emergy and emergy analysis since the 1980's. Even if there are attractive features, as mentioned in Hau and Bakshi [Section 1 and subsection 3.2],

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emergy analysis received many criticisms. Most of these criticisms could be applied to other popular methods which try to analyze environmental and industrial/human systems within the same framework. As mentioned in Hau and Bakshi (Section 5) emergy analysis of large and complex systems is one of the main challenges of the emergy approach. A system is large when it possesses a large number of components. A system is complex when there are splits and co-products within the same system. Generally speaking, an emergy system (see the precise definition in subsection 3.1) is represented by an oriented graph. Each node represents a process/component. The emergy circulates on the branches of the system and is assigned to the nodes of the system. A pathway from a source of emergy (e.g. sun, wind, fuel) on the graph, represents the sequel of assignations of the emergy source. According to Odum (1996) (Chap. 6, p. 90) in a split branching, a pathway of the emergy system is divided into several branches of the same kind e.g. as in hydraulic systems. In a co-product branching, the flow in each branch is of different kinds, e.g. in combined heat and power plants (described in e.g. Horlock (1996)).

The complexity comes from the fact that the flow circulating on a branch is in fact a combination of splits and co-products coming upstream this branch. And the upstream flows cannot be counted more than once. But it is clearly noticed in Lazzaretto (2009) (p.2201): "As observed by one of the reviewers the rule counting the largest emergy value (arriving at a node) is a rather "crude way" of avoiding double counting". The rules of emergy evaluation are explained in Chaper 6 (Odum, 1996). They are summarized in Sciubba and Ulgiati (2005) (pp. 1965 ~ 1966) as follows under the name emergy algebra：

R1: When only one product is obtained from a process (i.e. a process with only one output), all source-emergy is assigned to it.

Concerning processes with more than one output we have the following rules：

R2: When a flow (of emergy) splits the total emergy splits accordingly, based on the exergy/energy flowing through each pathway.

R3: When two or more co-products are generated in a process, the total source-emergy is assigned to each of them.

Finally, a fourth rule describes how emergy is assigned within a system of interconnected processes：

R4: Emergy cannot be counted twice within a system.

R4.1: Emergy in feedbacks cannot be double counted.

R4.2: Co-products, when reunited, cannot be summed. Only the emergy of the largest co-product flow is accounted for.

The general method of emergy analysis consists of propagating these rules from emergy sources to the outputs of the system of interconnected processes. Difficulty occurs for large and complex systems. To bypass this difficulty, several numerical methods have been proposed. Most of them are approximation methods based on linear algebra. Some of them are based on pre-analysis of the system which is not well-suited for an automatic emergy computation. For further information on such approaches see Li et al. (subsection 1.3 and references therein) (2010). Few simulation approaches have been proposed (see Odum and Peterson (1996), Maud (2007) and references therein). All these solutions have no mathematical framework and it is difficult to validate their results. To the best knowledge of the authors, the only mathematical framework which has been proposed in literature is Giannantoni (2006) who proposed another approach based on (non)-linear differential equations and on a variant of fractional derivatives concept.

In this paper, a method based on idempotent semiring, i.e. a semiring with an idempotent addition (for more details see Definition 2.1) is proposed. The starting point of this method is the Track summing method developed by Tennenbaum (1988) which is exact and has been implemented for emergy systems with splits and without co-products. More precisely the method starts from the expression given in Tennenbaum (p. viii) (1988) for acyclic source requirements. This formula expresses the emergy arriving at a node k of the system coming from all emergy sources upstream the node k avoiding the problem of double counting.

Contributions of the paper are as follows. First, we remark that the Tennenbaum's Track summing method can be divided into two different parts. The first part is a path-finding problem. Classification of paths for ecosystems appears necessary since Patten (1985). Then, the analysis of ecosystem network (or food webs) consists of the enumeration of paths and selection of these which belong to a certain class. In terms of formal language theory it is equivalent to extract words from the formal language associated with the graph which models the ecosystem network. In Whipple (1999) there are 16 categories of paths which are classified. The fourth rule of emergy algebra (no double assignation of emergy which occurs at the output of a node, see Brown and Herendeen (1996)) eliminates 14 categories of paths. Due to the specificity of the problem, the path-finding problem is solved by using idempotent semiring. It is a slight modification of methods enumerating elementary paths in a graph developed by Kaufmann and Malgrange (1963), Kaufmann (1967), Benzaken (1968), Backhouse and Carré (1975), also mentioned in Gondran and Minoux (2008). The second part is a computational problem. To completely solve the computational problem the axiomatic basis ($p_0 \sim p_6$) (see subsection 3.1), ($\omega.0 \sim \omega.2$), and ($\varphi.0 \sim \varphi.4$) (see subsection 3.3) is elaborated. This axiomatic basis is a translation of the abovementioned rules R1 ~ R4 (see the discussion in Section 6). Then, Definition 3.2 is proposed as the emergy measure. This definition and the axiomatic basis ($\omega.0 \sim \omega.2$), and ($\varphi.0 \sim \varphi.4$) allow us to provide an algorithm to compute emergy flowing on each arc of the emergy graph (see Section 4). In Proposition 4.1 the proposed algorithm is proved to terminate.

Organisation of the paper is based on the overall framework for computing emergy flowing on a given arc $[v_0; v_1]$ on an emergy graph G, see Figure 1. In Section 2 the basic material and main notations are presented to the reader. Idempotent semiring structure and elements of binary relations (especially the quotient of sets) are defined. Formal language and Max-Plus algebra are particular cases of idempotent semirings. They are detailed in subsections 3.1 and 3.3, respectively. In Section 3, elements of the method for computing emergy flowing on a given arc [v_0; v_1] of an emergy graph G are given. At the beginning of the method (i.e. block 1 in Figure 1) it is assumed that paths are modelled by words over a certain alphabet. It means that the emergy graph is represented by a formal language (which is a particular idempotent semiring). In subsection 3.1 the formal language F (see equation 17) associated with the emergy graph is given. The particularities of the emergy graph (see equation 18) are specified by axiom ($p_0 \sim p_6$) (i.e. block 2 in Figure 1). In subsection 3.2, the path-finding problem is formalized and solved (see Theorem 3.1). In definition 3.1, emergy path is defined. To extract from F the relevant words (i.e. emergy paths) a binary relation is defined by equations (22 ~ 26) and denoted $\Leftrightarrow_1 / \Leftrightarrow_2$. The block 3 in Figure 1 can be described as follows. If A_G^* denotes the matrix which contains all possible paths in the emergy graph G then the matrix $\bar{A}_G^*$ defined by equation (35) contains the emergy paths if the graph G. $\bar{A}_G^*$ is the result of the quotient of matrix A_G^* by the binary relation $\Leftrightarrow_1 / \Leftrightarrow_2$. $\bar{A}_G^*$ is computed step by step using particular sum, $\bar{\bigcup}$ defined by equation (33), and product $\bar{\bullet}$, defined by equation (34). The computation of the matrix $\bar{A}_G^*$ terminates (see the proof of theorem 3.1). In subsection 3.3, an axiomatic basis is ($\omega.0 \sim \omega.2$) and ($\varphi.0 \sim \varphi.4$) is proposed to provide a close formula for emergy flowing on an arc [v_0; v_1] of an emergy graph G (see block 4 in Figure 1). This formula is recursively obtained and expressed in the basic algebraic structure : the so-called max-plus algebra, defined by equation (37), which is a particular case of idempotent semiring. In Section 4 an algorithm is provided. It computes the emergy flowing on an arc accordingly to the Definition 3.2 of the emergy measure. In Proposition 4.1 the algorithm is proved to terminate. Note that the step [D] of the algorithm is recursive. In Section 5 the approach is validated by numerical examples. The subsection 5.1 is devoted to an emergy graph example with only splits. And one retrieves a similar formula as the Tennenbaum's one (the only difference is that the proposed formula is expressed using the max-plus algebra notations). The subsection 5.2 is devoted to an emergy graph with co-products only. Finally, an example of emergy graph with splits and co-products is analyzed in subsection 5.3. In Section 6 authors discuss rules R1 ~ R4 vs the axiomatic basis. Finally, Section 7 is the conclusion of the paper.

2. Main Definitions and Notations

2.1. Algebraic Structures

Let us define the fundamental (idempotent) algebraic structures used in this paper. The main reference is Baccelli et al. (1992) [Chapters 3 and 4]. The interested reader can also find many references in e.g. Glazek [subsection 4.2] (2002).

Definition 2.1 (Basic Structures):

- Magma: A magma is a set $\mathbb{M}$ equipped with an internal composition law, i.e. a map: $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$.
- Semigroup: A semigroup is a set $\mathbb{S}$ endowed with an associative operation $\oplus$: $\mathbb{S} \times \mathbb{S} \to \mathbb{S}$ (i.e. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$).
- Monoid: A monoid is a set M = ($\mathbb{M}$, $\oplus$, $\mathbf{0}$) which is a semigroup with a neutral element $\mathbf{0}$ (i.e. $a \oplus \mathbf{0} = \mathbf{0} \oplus a = a$). Moreover, if $\oplus$ is commutative (i.e. a $\oplus$ b = b $\oplus$ a) then M is a commutative monoid.
- Semiring: A semiring is a set S = ($\mathbb{S}$, $\oplus$, $\otimes$, $\mathbf{0}$, $\mathbf{1}$) with $\mathbf{0} \neq 1$ such that ($\mathbb{S}$, $\oplus$, $\mathbf{0}$) is a commutative monoid, $\otimes$: $\mathbb{S} \times \mathbb{S} \to \mathbb{S}$ is associative and its neutral element is $\mathbf{1}$, $\otimes$ has $\mathbf{0}$ as absorbing element (i.e. $\mathbf{0} \otimes a = a \otimes \mathbf{0} = \mathbf{0}$), $\otimes$ distributes over $\oplus$ (i.e. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ and $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$).

Semigroup, monoid, semiring are said to be idempotent when $\oplus$ is idempotent (i.e. $\forall a$, $a \oplus a = a$). Semirings are also known as dioids (see e.g. Bacecelli et al. (1992)).

Let $\mathrm{Mat}_n(\mathbb{S})$ be the set of $n*n$-matrices which entries are elements of S. If A $\in \mathrm{Mat}_n(\mathbb{S})$ we denote A(i, j), A(l, ·), A(·, k) the entry (i, j) of A, the l-row of A, the k-column of A, respectively. $(\cdot)^t$ denotes the transpose operator. The matrix ($\mathbf{0}$) denotes the null matrix.

The operations $\oplus$ and $\otimes$ are naturally extended on $\mathrm{Mat}_n(\mathbb{S})$ for all n > 1 as follows. Let A and B be two elements of $\mathrm{Mat}_n(\mathbb{S})$, then:

$$\mathbf{A} \oplus \mathbf{B} \stackrel{def}{=} \left(\mathbf{A}(i,j) \oplus \mathbf{B}(i,j)\right) \tag{1}$$

and

$$\mathbf{A} \otimes \mathbf{B} = \left(\mathbf{A}(i,\cdot) \otimes \mathbf{B}(\cdot,j)\right) = \left(\oplus_{k=1}^{n} \mathbf{A}(i,k) \otimes \mathbf{B}(k,j)\right) \tag{2}$$

The power function is defined as follow:

pow : $\mathbb{S} \setminus \{\mathbf{0}\} \times \mathrm{N} \to \mathbb{S} \setminus \{\mathbf{0}\}$

$$(s, n) \to \text{pow}(s, k) \stackrel{def}{=} \begin{cases} s \otimes \ldots \otimes s\ (k-\text{fold}) & \text{if } k \geq 1 \\ \mathbf{1} & \text{if } k = 0 \end{cases} \tag{3}$$

In the sequel pow(s, k) will be sometimes denoted $s^{\otimes k}$ or simply s^k. The power function is naturally extended to the set $\mathrm{Mat}_n(\mathbb{S})$ equipped with the above defined addition and multiplication of matrices as follows: $\mathrm{A}^{\otimes k} \stackrel{def}{=} \mathrm{A} \otimes \ldots \otimes \mathrm{A}$ (k-fold) if $k \geq 1$ and $\mathrm{A}^{\otimes 0} \stackrel{def}{=} \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, i.e. the matrix which off-diagonal entries are 0 and diagonal entries are $\mathbf{1}$.

We define the star/closure (or $*$-closure) of matrix A $\in \mathrm{Mat}_n(\mathbb{S})$ as:

$$\mathbf{A}^* \stackrel{def}{=} \mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^{\otimes k} \oplus \cdots, \text{ (if exists)} \tag{4}$$

Identifying $\mathrm{Mat}_1(\mathbb{S})$ with $\mathbb{S}$ the star/closure operator for scalars is well defined.

We summarize all these properties by saying that ($\mathrm{Mat}_n(\mathbb{S})$, $\oplus$, $\otimes$, $*$, ($\mathbf{0}$), $\mathbf{I}$) is a Kleene algebra, $\forall$ n >1.

2.2. Binary Relations

Let $\mathbb{X}$ and $\mathbb{Y}$ be two sets. In this paper it is sufficient to define a binary relation R over $\mathbb{X}$ and $\mathbb{Y}$ as a subset of the carte-

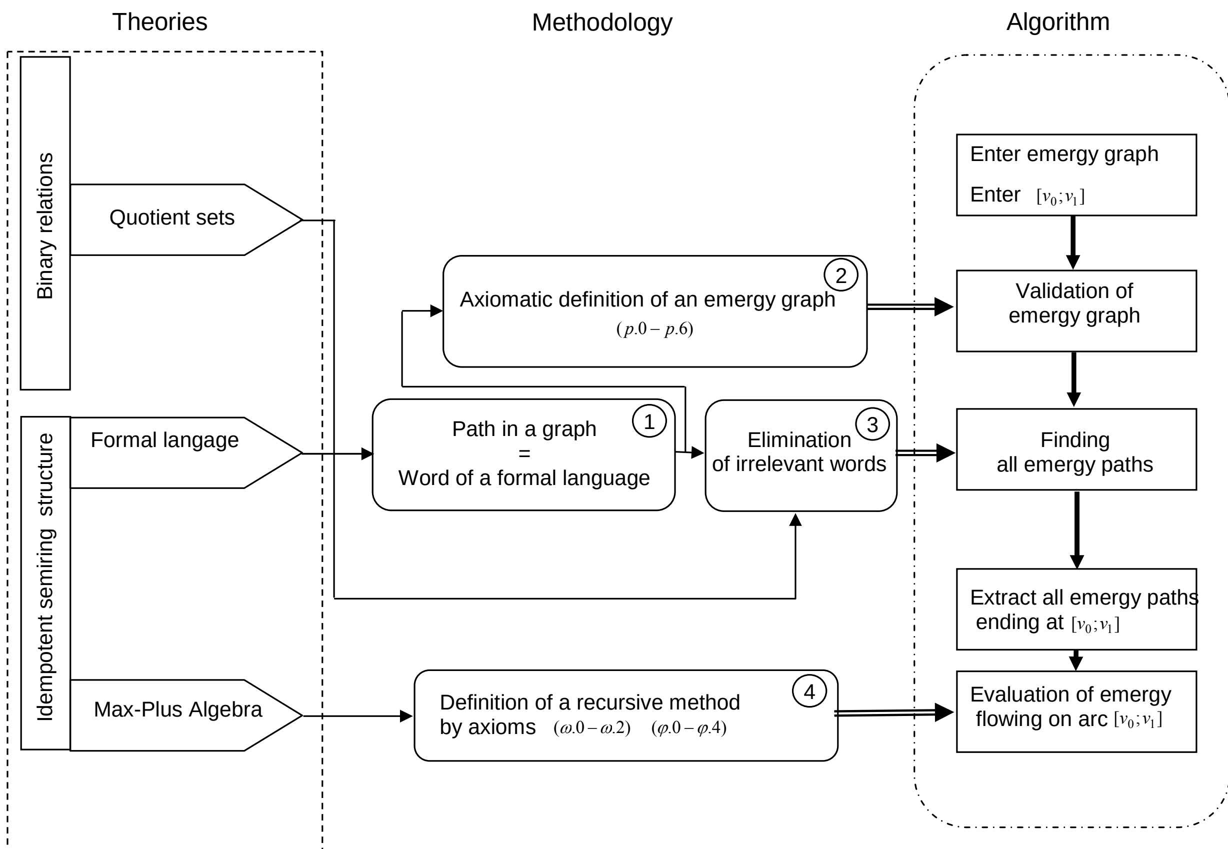

Figure 1. Overall framework.

sian product $\mathbb{X} \times \mathbb{Y}$. If $\mathbb{X} = \mathbb{Y}$ we simply say that R is defined over $\mathbb{X}$. The statement $(x, y) \in$ R is denoted $x\, R\, y$. The inverse relation of R, denoted R^{-1} is defined as $R^{-1} \overset{def}{=} \{(y,x) \in \mathbb{Y} \times \mathbb{X} \mid (x, y) \in R\}$. Let S be another binary relation over $\mathbb{X}$ and $\mathbb{Y}$ then the union of R and S denoted $R \cup S$ is defined as $R \cup S \overset{def}{=} \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid (x,y) \in R \text{ or } (x, y) \in S\}$. Let $\mathbb{Z}$ be a third set. Let T be a binary relation over $\mathbb{Y}$ and $\mathbb{Z}$. Then, the composition relation $R \circ T$ is a binary relation over $\mathbb{X}$ and $\mathbb{Z}$ defined as $R \circ T \overset{def}{=} \{(x,z) \in \mathbb{X} \times \mathbb{Z} \mid \exists y \in \mathbb{Y}, s.t.\ (x,y) \in R \text{ or } (y, z) \in T\}$, where the abbreviation s.t. means "such that".

Let R be a relation defined over the set $\mathbb{X}$. R is reflexive if $\forall x \in \mathbb{X}$, $x\, R\, x$. R is symmetric if $\forall x, x' \in \mathbb{X}$, $x\, R$ x' and $x'\, R\, x$. R is transitive if $\forall x, x', x'' \in \mathbb{X}$, $x\, R\, x'$ and $x'\, R\, x''$ implies $x\, R\, x''$. A relation which is reflexive and transitive is a preorder. A preorder which is also symmetric is called an equivalence relation.

If R is an equivalence relation over the set $\mathbb{X}$. The equivalent class of an element $a \in \mathbb{X}$ by the relation R is the subset of $\mathbb{X}$ defined by:

$$\mathrm{cl}_R(a) \overset{def}{=} \{\, x \in \mathbb{X} \mid x\, R\, a\} \tag{5}$$

If the context is clear the subscript R will be omitted. Thus, cl_R will be simply denoted as cl.

The quotient set of $\mathbb{X}$ by R is the set of all equivalent classes and is denoted $\mathbb{X} / R$. If S is another equivalent relation on $\mathbb{X} / R$, we define the relation R / S over $\mathbb{X}$:

$$\forall a,b \in \mathbb{X},\ a\, R / S\, b \overset{def}{\Leftrightarrow} \mathrm{cl}_R(a)\, S\, \mathrm{cl}_R(b). \tag{6}$$

By the definition of R / S we have:

$$\mathrm{cl}_{R/S}(a) \overset{def}{=} \{x \in \mathbb{X} \mid \mathrm{cl}_R(a)\, S\, \mathrm{cl}_R(x)\}. \tag{7}$$

for all $a \in \mathbb{X}$. And we note that:

$$\mathbb{X} / (R / S) = (\mathbb{X} / R) / S \tag{8}$$

Let $(\mathbb{M}, \oplus)$ be a magma. Then a relation R over $\mathbb{M}$ is a congruence if it is an equivalent relation compatible with the internal composition law $\oplus$, i.e.:

$$\forall x,y,a,b \in \mathbb{M}, \left(x\, R\, y \Rightarrow a \oplus x \oplus b\, R\, a \oplus y \oplus b\right) \tag{9}$$

3. Emergy Analysis

This Section is devoted to the methodology for emergy computation, as depicted in Figure 1.

3.1. Emergy Description Model and Its Qualitative Axioms

The way by which emergy circulates in a multicomponent system is modelled by an oriented graph. The graph has input nodes called sources, intermediate nodes and output (or final) nodes. Each node is represented by an integer (i.e. an element of $\aleph$). Let us denote $\mathbb{L}^s$, $\mathbb{L}^i$ and $\mathbb{L}^o$ as the set of emergy sources, the intermediate nodes and the output nodes of the emergy graph, respectively. An arc is an element of the set $\mathbb{A} \subseteq \mathbb{L} \times \mathbb{L}$ with $\mathbb{L} \overset{def}{=} \mathbb{L}^s \cup \mathbb{L}^i \cup \mathbb{L}^o$, where ($\mathbb{L}^s$; $\mathbb{L}^i$; $\mathbb{L}^o$) is a partition of $\mathbb{L}$ and × denotes the cartesian product of sets. The set $\mathbb{A}$ verifies: $\mathbb{A} \cap \mathbb{L}^s \times \mathbb{L}^s = \mathbb{A} \cap \mathbb{L}^o \times \mathbb{L}^o = \emptyset$. It means that a source (resp. an output node) is not linked to another source (resp. an output node).

The drawing conventions for the emergy graph are depicted in Figure 2. A source is represented by the symbol Figure 2A, an intermediate node on the emergy graph is represented by Figure 2B, an output node is represented by Figure 2C. Splits are modelled by Figure 2D and co-products are modelled by Figure 2E.

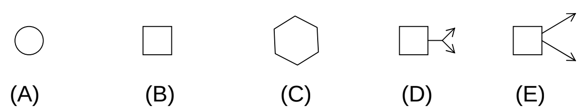

Figure 2. List of emergy symbols.

As the reader will see in the sequel it will be convenient to model emergy flows by particular words over a certain alphabet (block 1 in Figure 1). Thus, it is convenient to use some results of formal language theory. Formal laguages used here are particular cases of idempotent semirings.

Let us introduce the alphabet associated with the emergy graph:

$$\Sigma \overset{def}{=} \mathbb{L} \cup \{\underline{0}, \underline{1}, [,]\} \cup \{;\} \qquad (10)$$

The elements of Σ are called letters. Let us stress that the elements of $\mathbb{L}$ are the physical nodes of the emergy graph. The meaning of $\underline{0}$, $\underline{1}$ and ; is specified later in the paper.

The set of all words of finite length constructed on the alphabet Σ is the free monoid (see Bourbaki [A.I, pp. 77 ~ 79] (2006) for a very formal definition). It is defined as the following disjoint union (which coincides with the union of sets $\cup$):

$$\Sigma^* \overset{def}{=} \bigcup_{n \geq 0} \Sigma^n \qquad (11)$$

with $\Sigma^0 \overset{def}{=} \{\underline{1}\}$. It means that $\underline{1}$ is the empty word which has length 0. And $\forall\, n \geq 1$, Σ^n denotes the set of all words which contain exactly n letters. If $\mathbb{W}$ is any subset of Σ^* then: $\mathbb{W}_{\underline{0}} \overset{def}{=} \mathbb{W} \cup \{\underline{0}\}$. A word m is represented by a finite sequence of letters (i.e. elements of Σ).

On $\Sigma^*_{\underline{0}}$ two operations are defined. The first is the sum of two words which can be identified with the union if a word m is identified with the set $\{m\}$, and will then be denoted $\cup$. In other words it means that $\Sigma^*_{\underline{0}}$ will be identified with the set of all parts of $\Sigma^*_{\underline{0}}$, denoted $\mathbf{2}^{\Sigma^*_{\underline{0}}}$. The second operation is the concatenation of two words, denoted $\bullet$, which is defined as follows:

$$\begin{array}{llll} \bullet: & \Sigma^*_{\underline{0}} \times \Sigma^*_{\underline{0}} & \to & \Sigma^*_{\underline{0}} \\ & (m, m') & \mapsto & m \bullet m' \end{array} \qquad (12)$$

The word $m \bullet m'$ is the new word obtained by joining the letters of m and the letters of m' end-to-end. If there is no ambiguity the concatenated word $m \bullet m'$ will be denoted mm'.

Let us remark that

$$\mathrm{E} = (\Sigma^*_{\underline{0}}, \cup, \bullet, \underline{0}, \underline{1}) \qquad (13)$$

is an idempotent semiring (see section 2). The neutral element for $\bullet$ is $\underline{1}$ (the empty word). The element $\underline{0}$ (empty set) is the neutral element for $\cup$ and is absorbing for $\bullet$, i.e. $\forall\, m$, $\underline{0} \bullet m = m \bullet \underline{0} = \underline{0}$. The letter ; is used as a separator and is idempotent for concatenation, i.e.:

$$;\bullet; = ;. \qquad (14)$$

The set of idempotents for $\bullet$ is $\{\underline{0}, \underline{1}, ;\}$.

Let us also recall that the star/closure operator * is defined as follows. $\forall m \in \Sigma^*_{\underline{0}}$, $m^* \overset{def}{=} \bigcup_{n=0}^{\infty} m^n$, where $m^0 \overset{def}{=} \underline{1}$, $\forall n \geq 1$, $m^n \overset{def}{=} m, \ldots, m$ (n-fold). Note that $\forall m \in \Sigma^*_{\underline{0}}$, $\left(m^*\right)^* = m^*$ and $\underline{0}^* = \underline{1}^* = \underline{1}$.

The operations $\cup$, $\bullet$ and * are naturally extended to elements of $\mathrm{Mat}_n(\Sigma^*_{\underline{0}})$ and we summarize all above mentioned definitions by saying that $\left(\mathrm{Mat}_n\left(\Sigma^*_{\underline{0}}\right), \cup, \bullet, *, (\underline{0}), \mathbf{I}\right)$ is a Kleene algebra, $\forall n \geq 1$.

If $\mathbb{U}$ and $\mathbb{V}$ are two subsets of $\Sigma^*_{\underline{0}}$ the Minkowski product of $\mathbb{U}$ by $\mathbb{V}$:

$$\mathbb{U} \bullet \mathbb{V} \overset{def}{=} \{u \bullet v \mid u \in \mathbb{U}, v \in \mathbb{V}\} \overset{not.}{=} \mathbb{U}\mathbb{V}, \qquad (15)$$

is well defined because of the identification of $\Sigma^*_{\underline{0}}$ with $\mathbf{2}^{\Sigma^*_{\underline{0}}}$ and the distributivity of $\bullet$ over $\cup$. When $\mathbb{U}$ (resp. $\mathbb{V}$) is a singleton, i.e. $\mathbb{U} = \{m\}$ (resp. $\mathbb{V} = \{m'\}$), then $\mathbb{U} \bullet \mathbb{V}$ will be denoted $m \bullet \mathbb{V}$ (resp. $\mathbb{U} \bullet m'$) or simply $m\mathbb{V}$ (resp. $\mathbb{U}m'$). Since concatenation is associative the multiplication of sets is also associative. Let us also remark that $\forall n \geq 1$, $\Sigma^n = \Sigma \ldots \Sigma$ (n-fold).

We identify the set $\mathbb{L} \times \mathbb{L}$ with the set:

$$\mathbb{F} \overset{def}{=} [\bullet\mathbb{L}\bullet;\bullet\mathbb{L}\bullet] \overset{not.}{=} [\mathbb{L}; \mathbb{L}]. \qquad (16)$$

An emergy flow corresponds to a series of arcs of an emergy graph called physical emergy path in subsection 3.2. Thus, the main idempotent semiring describing flows of emergy in an emergy graph is:

$$F \overset{def}{=} (\mathbb{F}^*_{\underline{0}}, \cup, \bullet, \underline{0}, \underline{1}) \qquad (17)$$

The semiring F is a subsemiring (or sublanguage) of the semiring (or language) E.

One can now specify what is an emergy graph by using axioms ($p_0 \sim p_6$) (see block 2 in Figure 1). Let us consider four symmetric binary relations Ø, id, ⊥ and // defined on $\mathbb{A}$. For all $a, a' \in \mathbb{A}$: a Ø a' means that there is no relation between arcs a and a', the relation a id a' means $a = a'$ (identity relation over $\mathbb{A}$). For all $l, l', l'' \in \mathbb{L}$, $[l; l']$ // $[l; l'']$ means that there is a co-product at node l. And for all $l, l', l_1, l_2 \in \mathbb{L}$, $[l; l_1] \perp [l'; l_2]$ means that there is a split of emergy at node l if $l = l'$, or that l and l' are emergy sources. The binary relation ⊥ is called independent relation.

The relations Ø, id, ⊥ and // satisfy the following axioms:

- (p_0). For all arcs of the emergy graph there exists at least one binary relation between them, i.e. $\forall\, a, a' \in \mathbb{A}$, a Ø a' or a id a' or $a \perp a'$ or a // a'.
- (p_1). The binary relations ⊥ and // are in mutual exclusion, i.e. $\forall\, a, a' \in \mathbb{A}$, $a \perp a' \Rightarrow \neg(a \,//\, a')$. Where $\neg Q$ denotes the negation of the proposition Q.
- (p_2). If there exists a binary relation of type ⊥ or // between two arcs of the emergy graph necessarily these arcs are different,i.e. $\forall\, a, a' \in$ A, $a \perp a'$ or $a \,//\, a' \Rightarrow \neg(\, a$ id $a')$.
- (p_3). For $\dagger \in \{\text{id}, \perp, //\}$ we have the following pseudo-transitivity property: $\forall\, l \in \mathbb{L}$, $\forall\, a, a', a'' \in [l; \mathbb{L}] \cap \mathbb{A}$, $a \dagger a'$ and $a' \dagger a'' \Rightarrow a \dagger a''$.
- (p_4). There are only three possible binary relations between arcs of the emergy graph which have the same element of the set $\mathbb{L} \backslash \mathbb{L}^o$ as their input, i.e. $\forall\, l \in \mathbb{L} \backslash \mathbb{L}^o$, $\forall\, a, a' \in [l; \mathbb{L}] \cap \mathbb{A}$, a id a' or $a \perp a'$ or a // a'.
- (p_5). The different sources of an emergy graph are independent (cf. Odum, 1996), i.e. $\forall\, l, l' \in \mathbb{L}^s$, $\forall\, a \in [l; \mathbb{L}] \cap \mathbb{A}$, $\forall\, a' \in [l'; \mathbb{L}] \cap \mathbb{A}$, $a \perp a'$ or a id a'.
- (p_6). By convention, each source of the emergy graph is connected to only one node of the emergy graph, i.e. $\forall\, l \in \mathbb{L}^s$, $|[l; \mathbb{L} \setminus \mathbb{L}^s] \cap \mathbb{A}| = 1$.

At this step an emergy graph can be described by the following 10-tuple, called emergy graph G:

$$G \overset{def}{=} (\mathbb{L}, \mathbb{L}^s, \mathbb{L}^i, \mathbb{L}^o, F, \mathbb{A}, \text{id}, \perp, //, \text{Ø}) \tag{18}$$

recalling that ($\mathbb{L}^s$, $\mathbb{L}^i$ and $\mathbb{L}^o$) is a partition of $\mathbb{L}$, F defined by (17) is the formal language used to identify paths with words, $\mathbb{A} \subseteq [\mathbb{L}; \mathbb{L}]$ denotes the set of all arcs of the emergy graph which satisfies $\mathbb{A} \cap \mathbb{L}^s \times \mathbb{L}^s = \mathbb{A} \cap \mathbb{L}^o \times \mathbb{L}^o = \text{Ø}$, and id, ⊥, //, and Ø are the only possible binary symmetric relations between arcs of emergy graph G, which satisfies axioms ($p_0 \sim p_6$).

With the aim of computing emergy, the set of the arcs, $\mathbb{A}$, can be stored in the matrix $A_G \in Mat_{|\mathbb{L}|}(\mathbb{F}^*_O)$, where $|\mathbb{L}|$ denotes the number of elements of $\mathbb{L}$, called incidence matrix, and defined by:

$$\mathbf{A}_G(l, l') \overset{def}{=} \begin{cases} [l; l'], \textit{if } [l; l'] \in A \\ \underline{0}, \quad \textit{otherwise} \end{cases} \tag{19}$$

for all $l, l' \in \mathbb{L}$. The equality $\mathbf{A}_G(l, l') = [l; l']$ means that there exists one arc between node l and node l'. We label this arc $[l; l']$. We say that the graph G is a $\mathbb{F}$ -labelled graph.

The set of relations between the arcs of $\mathbb{A}$ is represented by a symmetric function $\mathbb{A} \times \mathbb{A} \rightarrow \{\text{id}, \perp, //, \text{Ø}\}$, which can be stored in an array $\mathbf{R}_G$ indexed by $\mathbb{A} \times \mathbb{A}$ with entries in {id, ⊥, //, Ø} by:

$$\mathbf{R}_G(l, l') \overset{def}{=} \dagger, \text{ if } a \dagger a', \tag{20}$$

where $\dagger \in \{\text{id}, \perp, //, \text{Ø}\}$. Let us note that the matrix $\mathbf{R}_G$ is filled up accordingly to the axioms ($p_0 \sim p_6$).

Example 3.1

Let us consider the emergy graph $G_1 = (\mathbb{L}_1, \mathbb{L}^s_1, \mathbb{L}^i_1, \mathbb{L}^o_1, F_1, \mathbb{A}_1, \text{id}, \perp, //, \text{Ø})$ of the Figure 3 which is an example borrowed from Brown and Herendeen [p.226, Figure 9b] (1996).

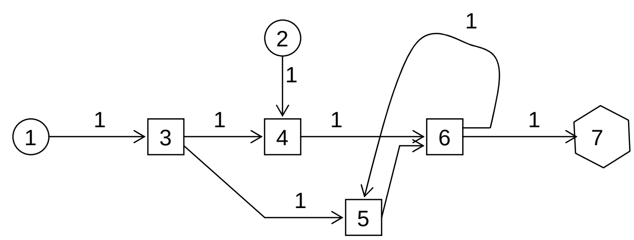

Figure 3. Graph G_1.

Table 1. Incidence Matrix, $\mathbf{A}_{G_1}$, of the Graph G_1

	1	2	3	4	5	6	7
1	0	0	[1;3]	0	0	0	0
2	0	0	0	[2;4]	0	0	0
3	0	0	0	[3;4]	[3;5]	0	0
4	0	0	0	0	0	[4;6]	0
5	0	0	0	0	0	[5;6]	0
6	0	0	0	0	[6;5]	0	[6;7]
7	0	0	0	0	0	0	0

Table 2. Array $\mathbf{R}_{G_1}$

$\mathbf{R}_{G_1}$	[1;3]	[2;4]	[3;4]	[3;5]	[4;6]	[5;6]	[6;5]	[6;7]
[1;3]	id	⊥	Ø	Ø	Ø	Ø	Ø	Ø
[2;4]	⊥	id	Ø	Ø	Ø	Ø	Ø	Ø
[3;4]	Ø	Ø	id	//	Ø	Ø	Ø	Ø
[3;5]	Ø	Ø	//	id	Ø	Ø	Ø	Ø
[4;6]	Ø	Ø	Ø	Ø	id	Ø	Ø	Ø
[5;6]	Ø	Ø	Ø	Ø	Ø	id	Ø	Ø
[6;5]	Ø	Ø	Ø	Ø	Ø	Ø	id	//
[6;7]	Ø	Ø	Ø	Ø	Ø	Ø	//	id

According to settings we have $\mathbb{L}_1$ ={1, 2, 3, 4, 5, 6, 7}, $\mathbb{L}^s_1$ = {1, 2}, $\mathbb{L}^i_1$ ={3, 4, 5, 6}, $\mathbb{L}^o_1$ ={7}. F_1 is defined by (17) with $\mathbb{F}$ replaced by $\mathbb{F}_1 = [\mathbb{L}_1; \mathbb{L}_1]$. The set of the arcs is $\mathbb{A}_1$ = {[1; 3], [2; 4], [3; 4], [3; 5], [4; 6], [5; 6], [6; 5], [6; 7]}. The incidence matrix, $\mathbf{A}_{G_1}$, of the graph G_1 is given in Table 1.

$\mathbf{A}_{G_1}(1, 3) = [1; 3]$ means that $[1; 3] \in \mathbb{A}_1$, $\mathbf{A}_{G_1}(2, 7) = \underline{0}$ means that $[2; 7] \notin \mathbb{A}_1$, and so on. Columns 1 and 2 of $\mathbf{A}_{G_1}$ are null columns, which mean that nodes 1 and 2 have no predecessors. The row 7 of $\mathbf{A}_{G_1}$ is a null row, which means that node 7 has no successors.

The relations between the arcs are stored in the array $\mathbf{R}_{G_1}$, see Table 2. It means equivalently that: $[1; 3] \perp [2; 4]$ because 1 and 2 are sources and axiom p_5. There are co-products at nodes 3 and 6. Thus, by definition of // we have: [3; 4] // [3; 5] and [6; 5] // [6; 7].

3.2. Emergy Flow Enumeration Problem

Let us introduce the following definitions from the semiring F defined by (17).

Definition 3.1.

- *Path*. A path π is an element of the set $\mathbb{F}_{\underline{0}}^*$ which has the form $\pi = \underline{0}$, or $\pi = \underline{1}$, or $\pi = [l_1; l_2]$ or $\forall\ k > 3$, $\pi = [l_1; l_2]\ [l_2; l_3] \ldots [l_{k-2}; l_{k-1}]\ [l_{k-1}; l_k]$, with $l_j \in \mathbb{L}$, $j = 1, \ldots, k$. The length of the path, lg(π), is $-\infty$ if $\pi = \underline{0}$, 0 is $\pi = \underline{1}$, otherwise the length of π is equal to the number of arcs $[l_j; l_{j+1}]$ which compose the path.
- *Path from a source*. A path from a source is a path $\pi = [l_1; l_2] \ldots [l_{k-1}; l_k]$, s.t. $l_1 \in \mathbb{L}^s$.
- *Simple path*. A simple path is a path $\pi = [l_1; l_2] \ldots [l_{k-1}; l_k]$, s.t. $\forall\ 1 < j \neq j' < k$, $l_j \neq l_{j'}$.
- *Simple path from a source*. A simple path from a source is a simple path $\pi = [l_1; l_2] \ldots [l_{k-1}; l_k]$, s.t. $l_1 \in \mathbb{L}^s$.
- *Emergy path*. An emergy path is a path $\pi = [l_1; l_2] \ldots [l_{k-1}; l_k]$, s.t. the path $[l_1; l_2] \ldots [l_{k-2}; l_{k-1}]$ is a simple path from a source and $l_k \in \mathbb{L} \backslash \mathbb{L}^s$.
- *A physical emergy path* (or *emergy flow*). A physical emergy path is an emergy path $\pi = [l_1; l_2] \ldots [l_{k-1}; l_k]$ s.t. $\forall 2 < j < k - 1$, $l_j \in \mathbb{L}^i$ and $l_k \in \mathbb{L} \backslash \mathbb{L}^s$.

In fact, because of the structure of an emergy graph, an emergy path defined on an emergy graph coincides with a physical emergy path.

Let us illustrate the different notions of paths previously defined, see Figure 3. A path π has the form $\pi = 0$ (impossible path or empty set), $\pi = 1$ (unit path, i.e. a path with no arc) or e.g. π = [3; 4] [4; 6] [6; 5] [5; 6] which is a path from first node 3 to last node 6 in graph G_1. A path from a source is a path such that its first node is a source, e.g. [1; 3] [3; 4] [4; 6] [6; 5] [5; 6] [6; 5] is a path from the source 1 to node 5. A simple path is a path such that all its nodes are different, e.g. [4; 6] [6; 5] is a simple path from node 4 to node 5. A simple path from a source is a simple path such that its first node is a source, e.g. [2; 4] [4; 6] [6; 5] is a simple path from the source 2 to node 5. The path [2; 4] [4; 6] [6; 5] and [2; 4] [4; 6] [6; 5] [5; 6] are emergy paths. Note that the loop 6; 5; 6 is only counted once. But the path [2; 4] [4; 6] [6; 5] [5; 6] [6; 7] is not an emergy path because the path [2; 4] [4; 6] [6; 5] [5; 6] is not a simple path from a source. Let us stress that this definition of emergy path solves the problem of the rule R4.1 (i.e. Emergy in feedbacks cannot be double counted).

For all $k > 2$, for all $l' \in \mathbb{L} \backslash \mathbb{L}^s$, let $\mathbb{S}_k$, $\mathbb{S}_{s,k}$, and $\mathbb{E}_k(l')$ be the set of all simple paths of length k, the set of all simple paths from a source of length k and the set of all emergy paths of length k ending at node l', respectively. Then, we have:

$$\mathbb{S}_{s,k} = [\mathbb{L}^s; \mathbb{L} \backslash \mathbb{L}^s]\ \mathbb{S}_{k-1},\ \mathbb{E}_k(l') = \mathbb{S}_{s,k-1}\ [\mathbb{L} \backslash \mathbb{L}^s; l']. \tag{21}$$

From these trivial relations between the sets it is clear that the emergy flow enumeration problem is very close to a simple path enumeration problem. Such problems have already been studied in the literature (see Benzaken (1968), Backhouse and Carré (1975) and references therein).

To solve the path-finding problem (i.e. block 3 of the methodology depicted in Figure 1), let us define the following binary relations (or rewriting rules) denoted $\rightarrow_1$ and $\rightarrow_2$, respectively defined by:

$$\forall l \in \mathbb{L} : l][l \rightarrow_1 l. \tag{22a}$$

And

$$\forall l \in \mathbb{L}_{\underline{0}},\ \forall m \in \mathbb{F}^*,\ \forall l' \in \mathbb{L} : l;\ m;\ l;\ l' \rightarrow_2 \underline{0}. \tag{22b}$$

The relation $\rightarrow_1$ is used to list the nodes through which emergy passes. The relation $\rightarrow_2$ is used to detect if there is a double assignation of emergy at a node l.

We follow Book and Otto [Chapters 1, 2 and 7] (1993) to construct the Thue congruences $\Leftrightarrow_1$ and $\Leftrightarrow_2$ associated with $\rightarrow_1$ and $\rightarrow_2$, respectively. The Thue congruence $\Leftrightarrow_1$ transforms any path of the emergy graph into the series of nodes which have assigned the emergy following this path. Then, the Thue congruence $\Leftrightarrow_2$ applied to sequel of nodes which have assigned the emergy detects and eliminates sequence of nodes which contains a motive of the form $l;\ m;\ l;\ l'$ with $l' \neq \underline{0}, \underline{1}$ (i.e. emergy at node l is assigned at least twice). In other words, we apply the quotient relation $\Leftrightarrow_1 / \Leftrightarrow_2$ to any path of the emergy graph to detect and eliminate double assignation problem. In order to make this paper self-contained the main steps of the construction of the relations $\Leftrightarrow_i$, i = 1, 2, are given hereafter.

a) We define the single-step rewriting relations $\Rightarrow_1$ and $\Rightarrow_2$ respectively induced by $\rightarrow_1$ and $\rightarrow_2$, as follows:

$$s \Rightarrow_i t \overset{def}{\Leftrightarrow} \exists\, x, y, u, v \in \textstyle\sum^* : s = xuy,\ t = xvy \text{ and } u \rightarrow_i v, \text{ for } i = 1, 2. \tag{23}$$

b) We define for i = 1, 2:

$$\Leftarrow_i \overset{def}{=} \Rightarrow_i^{-1}, \tag{24}$$

$$\Leftrightarrow_i \overset{def}{=} \Rightarrow_i \cup \Leftarrow_i, \tag{25}$$

$$\overset{*}{\Leftrightarrow}_i \overset{def}{=} \cup_{n \in \aleph} \Leftrightarrow_i^n, \tag{26}$$

with the convention: $\Leftrightarrow_i^0 \overset{def}{=}$ id (the identity relation). For all $\xi \in \mathbb{F}_{\underline{0}}^*$ the equivalence class $\mathrm{cl}_{\overset{*}{\Leftrightarrow}_1 / \overset{*}{\Leftrightarrow}_2}(\xi)$ will be denoted $\mathrm{cl}_{1/2}(\xi)$.

Remark 3.1. Replacing $\rightarrow 2$ by:

$$\forall\, l \in \mathbb{L}_{\underline{0}},\ \forall m \in \mathbb{F}^* : l;\ m;\ l \rightarrow_3 \underline{0}. \tag{27}$$

one obtains the gerbier of null square as mentioned in Benzaken (pp. 53 ~ 56) (1968) which has been successfully utilized to enumerate elementary paths on a graph. See also Backhouse and Carré (pp. 182 ~ 183) (1975) and references therein.

Binary relations, necessary to eliminate irrelevant words (or paths), see block 3 in Figure 1, have been introduced. At first sight, one possible approach could be the enumeration of all paths of emergy graph by computing the *-closure of the incidence matrix of the graph, A_G^*. Then, for all nodes i and j of the graph, compute the quotient set $A_G^* A_G^*(i,j) \big/ \overset{*}{\Leftrightarrow}_1 \big/ \overset{*}{\Leftrightarrow}_2$, recalling that $A_G^*(i,j)$ is the set of all paths from i to j. The major drawback of this direct method is that there exist loops in the graph and hence element of $A_G^*(i,j)$ can be paths with infinite length. To bypass this problem operator $\overline{}$, sum $\overline{\cup}$, and product $\bar{\bullet}$ are introduced in what follows. Let us begin by the following remark.

Remark 3.2. For all $\sigma \in \mathbb{F}^*$, $\mathrm{cl}_{1/2}(\sigma)$ has only two values:

- $\mathrm{cl}_{1/2}(\sigma) = \{\underline{0}\}$ which is equivalent to say that $\sigma = m\ \mu\ m'\ \mu\ m''$ where $\mu, m, m' \in \mathbb{F}^*$ and $m'' \notin \{\underline{0}, \underline{1}\}$.
- $\mathrm{cl}_{1/2}(\sigma) = \{\sigma\}$ which is equivalent to say that σ has no sequence of the form $m\ \mu\ m'\ \mu\ m''$ where $\mu, m, m' \in \mathbb{F}^*$ and $m'' \notin \{\underline{0}, \underline{1}\}$.

From the above Remark 3.2 we define the operator $\overline{}$: $\mathbb{F}_{\underline{0}}^* \rightarrow \mathbb{F}_{\underline{0}}^*$ as follows:

$$\forall m \in \mathbb{F}_{\underline{0}}^*\ \ \overline{m} \overset{def}{=} \begin{cases} \underline{0} & \text{if } \mathrm{cl}_{1/2}(m) = \{\underline{0}\} \\ m & \text{otherwise.} \end{cases} \tag{28}$$

For all set $\mathbb{U} \subseteq \mathbb{F}_{\underline{0}}^*$ we define:

$$\overline{\mathbb{U}} \overset{def}{=} \left\{ \overline{u} \mid u \in \mathbb{U} \right\}. \tag{29}$$

This is equivalent to impose that the operator $\overline{}$ satisfies $\overline{m \cup m'} = \overline{m} \cup \overline{m'}$.

From the operator $\overline{}$ we define two new binary operations $\bar{\bullet}$, $\overline{\cup} : \mathbb{F}_{\underline{0}}^* \times \mathbb{F}_{\underline{0}}^* \rightarrow \mathbb{F}_{\underline{0}}^*$ s.t. for all $m, m' \in \mathbb{F}_{\underline{0}}^*$:

$$m \,\bar{\bullet}\, m' \overset{def}{=} \overline{m \bullet m'},\ m \,\overline{\cup}\, m' \overset{def}{=} \overline{m \cup m'} = \overline{m} \cup \overline{m'} \tag{30}$$

Proposition 3.1. For all $m, m_1, m_2, m_3, m_4 \in \mathbb{F}_{\underline{0}}^*$ we have:

$$\overline{\overline{m}} = \overline{m}\,. \tag{31}$$

$$m_1 \,\bar{\bullet}\, m_2 \,\overline{\cup}\, m_3 \,\bar{\bullet}\, m_4 = \overline{m_1 \bullet m_2 \cup m_3 \bullet m_4} \tag{32}$$

Proof:

To prove (31) we just have to note that $\overline{\underline{0}} = \underline{0}$ because $\mathrm{cl}_{1/2}(\underline{0}) = \{\underline{0}\}$. Let us prove (32). Based on (30) we have:

$$\begin{aligned} m_1 \,\bar{\bullet}\, m_2 \,\overline{\cup}\, m_3 \,\bar{\bullet}\, m_4 &= \overline{m_1 \,\bar{\bullet}\, m_2 \cup m_3 \,\bar{\bullet}\, m_4} \\ &= \overline{m_1 \,\bar{\bullet}\, m_2} \cup \overline{m_3 \,\bar{\bullet}\, m_4} \\ &= \overline{\overline{m_1 \bullet m_2}} \cup \overline{\overline{m_3 \bullet m_4}} \quad \text{by (31).} \\ &= \overline{m_1 \bullet m_2} \cup \overline{m_3 \bullet m_4} \\ &= \overline{m_1 \bullet m_2 \cup m_3 \bullet m_4} \end{aligned}$$

□

The binary operations $\overline{\cup}$ and $\bar{\bullet}$ are naturally extended to the elements of $\mathrm{Mat}_n(\mathbb{F}_{\underline{0}}^*)$ as follows. Let A and B be two elements of $\mathrm{Mat}_n(\mathbb{F}_{\underline{0}}^*)$, then:

$$\mathbf{A} \,\overline{\cup}\, \mathbf{B} \overset{def}{=} \left(\mathbf{A}(i,j) \,\overline{\cup}\, \mathbf{B}(i,j) \right) \tag{33}$$

and

$$\mathbf{A} \,\bar{\bullet}\, \mathbf{B} \overset{def}{=} \left(\mathbf{A}(i,\cdot) \,\bar{\bullet}\, \mathbf{B}(\cdot,j) \right) = \left(\overline{\bigcup}_{k=1}^{n} \mathbf{A}(i,k) \,\bar{\bullet}\, \mathbf{B}(k,j) \right) \tag{34}$$

Remark 3.3. By definition of $\overline{\cup}$, $\bar{\bullet}$ and Proposition 3.1 we compute matrices $\mathbf{X} = \mathbf{A} \,\overline{\cup}\, \mathbf{B}$ and $\mathbf{Y} = \mathbf{A} \,\bar{\bullet}\, \mathbf{B}$ as follows:

1. $\mathbf{X} := \mathbf{A} \cup \mathbf{B} = \left(\mathbf{A}(i,j) \cup \mathbf{B}(i,j) \right)$
2. $\mathbf{X} := \overline{\mathbf{X}}$

and:

1'. $\mathbf{Y} := \mathbf{A} \bullet \mathbf{B} = \left(\bigcup_{k=1}^{n} \mathbf{A}(i,k) \bullet \mathbf{B}(k,j) \right)$

2'. $\mathbf{Y} = \overline{\mathbf{Y}}$

where $\overline{\mathbf{X}} \overset{def}{=} \left(\overline{\mathbf{X}(i,j)} \right)$ and $\overline{\mathbf{Y}} \overset{def}{=} \left(\overline{\mathbf{Y}(i,j)} \right)$.

Remark 3.4. Let us note that the product $\bar{\bullet}$ is a slight modification of the so-called latin multiplication (see Kaufmann and Malgrange (1963), Kaufmann (Chap. IV, Section 39) (1967), see also Gondran and Minoux (Chap. 4, Section 6.2) (2008).

For all matrix $\mathrm{A} \in \mathrm{Mat}_n(\mathbb{F}_{\underline{0}}^*)$ we define $\mathbf{A}^{\bar{\bullet}k} \overset{def}{=} \mathbf{A} \,\bar{\bullet} \cdots \bar{\bullet}\, \mathbf{A}$ (k-fold), if $k > 1$, $\mathbf{I}$ if $k = 0$. Finally, let us define the *-closure of the matrix A, $\mathbf{A}^{\bar{*}}$, by:

$$\mathbf{A}^{\bar{*}} \overset{def}{=} \overline{\bigcup}_{k \geq 0} \mathbf{A}^{\bar{\bullet}k} \tag{35}$$

Based on the definition of $\overline{\cup}$ and $\bar{\bullet}$ it is not difficult to see that the matrices $\mathbf{A}^{\bar{\bullet}k}$ and $\mathbf{A}^{\bar{*}}$ have the following interpretation.

Remark 3.5. The term $\mathbf{A}^{\bar{\bullet}k}(i,j)$ is the set of all emergy flows of length k from node i to node j. The term $\mathbf{A}^{\bar{*}}(i,j)$ is the set of all emergy flows from node i to node j.

Theorem 3.1. (Emergy flow enumeration) Assume the axioms ($p_0 \sim p_6$). Let us consider an emergy graph $G = (\mathbb{L}, \mathbb{L}^s, \mathbb{L}^i, \mathbb{L}^o, F, \mathbb{A}, \mathrm{id}, \perp, //, \varnothing)$ defined by (18). Let us consider its incidence matrix, $\mathbf{A}_G$, defined by (19) and its relation array between the arcs of $\mathbb{A}$, $\mathbf{R}_G$, defined by (20). Then, we have:

(a) The matrix $\mathbf{A}_G^{\bar{*}}$, defined by (35), exists.

(b) The set of all physical emergy paths coincides with the set of emergy paths:

$$\varepsilon \overset{def}{=} \bigcup_{l \in L^s} \mathbf{A}_G^{\bar{*}}(l,\cdot) \tag{36}$$

recalling that $\mathbf{A}_G^{\bar{*}}(l,.)$ denotes the l-row of the matrix $\mathbf{A}_G^{\bar{*}}$.

Proof. The result (a) is a classical result. In fact, developing similar arguments (i.e. Lunc Theorem and Remark 3.5) as in Benzaken [Proposition p. 53] (1968) we easily prove that: $\forall k > |\mathbb{L}^i| + 2, \mathbf{A}^{\bar{\bullet}k} = (\underline{0})$. The result (b) is a straightforward consequence of the interpretation of the entries (i, j) of the matrix $\mathbf{A}_G^{\bar{*}}$ (see Remark 3.5) and the definition of an emergy flow (see Definition 3.1). □

3.3. Emergy Evaluation

In this subsection it is assumed that the set of all emergy paths ε, defined by (36), is given. The block 4 in Figure 1 is now detailed. Lets us introduce the following idempotent semiring called Max-plus algebra:

$$S = (\mathbb{S} \stackrel{def}{=} \Re_+ \cup \{-\infty\}, \oplus \stackrel{def}{=} \max, \otimes \stackrel{def}{=} +, \mathbf{0} \stackrel{def}{=} -\infty, \mathbf{1} \stackrel{def}{=} 0), \quad (37)$$

where $\Re_+$ denotes the set of nonnegative reals. max denotes the maximum and + is the usual addition on $\Re$ (set of real numbers). $\oplus \stackrel{def}{=} \max$ means that $a \oplus b \stackrel{def}{=} \max(a, b)$. $\otimes \stackrel{def}{=} +$ means that $a \otimes b \stackrel{def}{=} a+b$.

On the max-plus algebra S the power function is s.t. $\forall \in \mathbb{S}$, $\forall n \in \aleph$, $\mathrm{pow}(s, n) = s \cdot n$ where $\cdot$ denotes the multiplication on $\Re$. In this context the power function can be naturally extended to the power function denoted POW as follows:

$$\begin{array}{lll} \mathrm{POW}: \mathbb{S}\backslash\{\mathbf{0}\} \times \mathbb{S}\backslash\{\mathbf{0}\} & \rightarrow & \mathbb{S}\backslash\{\mathbf{0}\} \\ (s,s') & \mapsto & \mathrm{POW}(s, s') \stackrel{def}{=} s \cdot s'. \end{array} \quad (38)$$

Let us remark that POW is symmetric (i.e. POW(s, s') = POW (s',s)) and associative (i.e. POW(s, POW(s', s'')) = POW(POW (s, s'), s'') = POW(s, s', s'')).

Let us recall that $\mathbb{L}$ denotes the set of all nodes of the emergy graph G. The emergy graph G is defined by (18). We define the following functions. The emergy function $\theta : \mathbb{L} \rightarrow \Re_+$ s.t. $\forall\ l \notin \mathbb{L}^s$, $\theta(l)$=0. Emergy function attributes emergy to the sources of the graph G. The weight function $\omega : \mathbb{F}^* \rightarrow \Re_+$ which satisfies the following axioms:

- (ω.0). $\omega(\underline{1})$=1.
- (ω.1). $\omega([l; l'])$ corresponds to the fraction of emergy (which is assumed to be given in this paper) circulating on $[l; l']$ if $[l; l']$ is an arc of the emergy graph, 0 otherwise.
- (ω.2). $\omega(m \bullet m') = \omega(m) \cdot \omega(m') = \mathrm{POW}(\omega(m), \omega(m'))$. It means that the function ω is a (•,·)-morphism.

Until now in this paper, $\sum_{\underline{0}}^* (\mathbb{F}_{\underline{0}}^*)$ was identified with $\mathbf{2}^{\Sigma_{\underline{0}}^*} (2^{\mathbb{F}_{\underline{0}}^*})$ but at this step of the paper it is clearer to make the distinction between $\sum_{\underline{0}}^* (\mathbb{F}_{\underline{0}}^*)$ and $\mathbf{2}^{\Sigma_{\underline{0}}^*} (2^{\mathbb{F}_{\underline{0}}^*})$. Let us define the set function $\varphi: \mathbf{2}^{\Sigma_{\underline{0}}^*} \rightarrow \Re_+$ which allows us to compute the emergy flowing on every arc of the emergy graph. The function φ satisfies the following axioms:

- (φ. 0). $\varphi(\underline{1}) = 1$, $\varphi(\underline{0}) = 0 \stackrel{def}{=} \varphi(\varnothing)$.
- (φ. 1). $\forall\ m \in \mathbb{F}_{\underline{0}}^*$, $\varphi(m) = \varphi(\{m\})$.
- (φ. 2).

$$\varphi([l; l']) = \begin{cases} \omega([l; l']) & \text{if } l, l' \notin \mathbb{L}^s \\ \mathrm{POW}(\theta(l), \omega([l; l'])) & \text{if } l \in \mathbb{L}^s \text{ and } l' \notin \mathbb{L}^s \\ 0 & \text{otherwise} \end{cases}$$

- (φ.3). $\forall\ m \in \mathbb{F}^*$, $\forall\ \mathbb{U} \in \mathbf{2}^{\Sigma_{\underline{0}}^*}$, $\varphi(m\ \mathbb{U}) = \mathrm{POW}(\varphi(m), \varphi(\mathbb{U}))$
- (φ.4). Let us consider the situation where the quantity flows have a common upstream flow m and are reunited at arc $[l; l']$ after a split (⊥) or a co-product (//). That is $\forall\ m \in \mathbb{F}^*$, $\forall\ a_1, ..., a_k \in \mathbb{A}$ s.t. $a_1 \dagger a_2, ..., \dagger a_k$ with $\dagger \in \{\perp, //\}$, $\forall\ \mathbb{U}_1, ..., \mathbb{U}_k \in \mathbf{2}^{\Sigma_{\underline{0}}^*}$:
- (φ.4.1). If the arcs a_i are linked by the relation ⊥ then the total quantity flowing on arc $[l; l']$ $\varphi\left(\bigcup_{i=1}^k m\ a_i\ \mathbb{U}_i\right)$ is equal to the sum of the quantities flowing on arc $[l; l']$ of the system as if there was only one arc a_i after the upstream flow m, $\varphi(m\ a_i\ \mathbb{U}_i)$, $i = 1, ..., k$, when reunited, i.e.

$$\varphi\left(\bigcup_{i=1}^k m\ a_i\ \mathbb{U}_i\right) = \otimes_{i=1}^k \varphi\left(m\ a_i\ \mathbb{U}_i\right) \text{ if } \dagger = \perp.$$

(See the explanation in Appendix A).

- (φ.4.2). If there are co-products just after m then the total quantity flowing on arc $[l; l']$, $\varphi\left(\bigcup_{i=1}^k m\ a_i\ \mathbb{U}_i\right)$, is equal to the maximum of the quantities flowing on arc $[l; l']$ of the system as if there was only one arc a_i after the upstream flow m, $\varphi(m\ a_i\ \mathbb{U}_i)$, $i = 1,..., k$, when reunited, i.e.:

$$\varphi\left(\bigcup_{i=1}^k m\ a_i\ \mathbb{U}_i\right) = \oplus_{i=1}^k \varphi\left(m\ a_i\ \mathbb{U}_i\right) \text{ if } \dagger = //.$$

(See the explanation in Appendix B).

We call (φ.3) ~ (φ.4) the tree property. And we are now in position to define the emergy measure flowing on an arc of an emergy graph.

Definition 3.2 (Emergy measure) Assume the axioms (p_0 ~ p_6). Assume also the axioms (ω.0 ~ ω.2), (φ.0 ~ φ.2) and the tree property (φ.3 ~ φ.4). Let us consider the emergy graph $G = (\mathbb{L}, \mathbb{L}^s, \mathbb{L}^i, \mathbb{L}^o, F, \mathbb{A}, \mathrm{id}, \perp, //, \varnothing)$ defined by (18). Let us consider its incidence matrix, $\mathbf{A}_G$, defined by (19) and its relation array between the arcs of $\mathbb{A}$, $\mathbf{R}_G$, defined by (20). Then, the emergy flowing on arc $[l; l']$ with $l, l' \in \mathbb{L}$, Em ($[l; l']$), is defined by:

$$\mathrm{Em}([l; l']) \stackrel{def}{=} \varphi(\varepsilon([l; l'])), \quad (39)$$

where $\varepsilon([l; l'])$ denotes the subset of the elements of the set ε (defined by (36, Theorem 3.1)) ending by $[l; l']$.

4. Algorithm for Emergy Computation

In this Section we present a recursive algorithm to compute Em($[v_0; v_1]$) which is as follows (see Figure 1):

Enter G, $\mathbf{A}_G$, $\mathbf{R}_G$, v_0 and v_1.

[A] Validation of the graph G based on (p_0 ~ p_6).

[B] Compute $\mathbf{A}_G^{\bar{*}}$ (see equation 35):

$\mathbf{X} := \mathbf{I}$, $\mathbf{S} := \mathbf{I}$

Do

$\mathbf{X} := \underline{\mathbf{A}}_G \bullet \mathbf{X}$

$\mathbf{X} := \overline{\mathbf{X}}$, recalling $\overline{\;\;}$ is defined by (28)

$\mathbf{S} := \mathbf{S} \bigcup \mathbf{X}$

Until $\mathbf{X} = (\underline{0})$

recalling that $(\underline{0})$ denotes the null matrix

[C] Compute $\varepsilon([v_0; v_1])$ the subset of the elements of set ε (defined by (36, Theorem 3.1)) ending by $[v_0; v_1]$.

[D] Compute $\varphi(\varepsilon([v_0; v_1]))$ (see Definition 3.2) recursively. Let Ψ be the current set (initially $\Psi = \varepsilon([v_0; v_1])$, we have:

While $\Psi \# \varnothing$, do the following:

1. Factorize Ψ according to (φ.4) using the same notations.
2. Apply (φ.4.0) if † = id or (φ.4.1) if † = ⊥ or (φ.4.2) if † = //.
3. Apply (φ.3) to each $m\, a_i\, \mathbb{U}_i$, i=1, …, k if † ∈ {⊥, //} or Apply (φ.3) to $m\, a_1\, \left(\bigcup_{i=1}^{k} \mathbb{U}_i\right)$ if † = id.
4. $\Psi := \Psi \setminus \left(\bigcup_{i=1}^{k} \{m\, a_i\}\right)$.

End.

Return Em($[v_0; v_1]$).

Proposition 4.1. The previous algorithm terminates.

Proof. Based on the result of Theorem 3.1 the steps [B] and [C] of the algorithm terminate trivially. Thus, it remains to prove that the recursive step [D] also terminates.

Initially, $\Psi = \varepsilon([v_0; v_1])$. By definition of set $\varepsilon([v_0; v_1])$ and axiom (p_6) all elements are of the form: $[l_i, l'_i]\, w_{ij}$, with $l_i \in \mathbb{L}^s$, $l'_i \in \mathbb{L} \setminus \mathbb{L}^s$, $w_{ij} \in \mathbb{F}^*[v_0; v_1]$, $i = 1, \ldots, k, j = 1, \ldots, q(i)$, for some $2 < k < |\mathbb{L}^s|$ (note that the case k=1 is trivial) and $q(i) > 1$. Then, we write:

$$\Psi = m\, a_1\, \mathbb{U}_1 \;\bigcup\; \ldots \;\bigcup\; m\, a_k\, \mathbb{U}_k$$

with $m = \underline{1}$, $\mathbb{U}_i = \bigcup_{j=1}^{q(i)} \{w_{ij}\}$ and by axiom (p_5) we have: $a_1 = [l_1; l'_1] \perp a_2 = [l_2; l'_2] \perp \ldots \perp a_k = [l_k; l'_k]$.

Thus, we can apply (φ.4.1) and (φ.3) to each $m\, a_i\, \mathbb{U}_i$, $i = 1, \ldots, k$. And hence, it remains to prove that the sets $\mathbb{U}_i$ can be decomposed accordingly to the axiom (φ.4).

Let us first remark that elements of $\mathbb{U}_i$ begin by $[l'_i$ by definition of a path in a graph. Then, by axioms ($p_1 \sim p_4$) there is only one possible relation † ∈ {⊥, //, id} between the arcs starting from $[l'_i$. Thus, once again it is possible to factorize $\mathbb{U}_i$ according to the axiom (φ.4). Based on the same arguments already used, we recursively decompose at each step of the algorithm the sets of the form $\mathbb{U}_i$ until $\mathbb{U}_i = \varnothing$.□

5. Numerical Examples

5.1. Tennenbaum-like Example

Let us consider the emergy graph G_2 corresponding to the Figure 4. $\mathbb{L}_2 = \{1, 2, 3, 4, 5\}$, $\mathbb{L}_2^s = \{1, 2\}$, $\mathbb{L}_2^i = \{3,4\}$, $\mathbb{L}_2^o = \{5\}$. $\mathbb{A}_2 = \{[1; 3], [2; 4], [3; 4], [4; 3], [4; 5]\}$. Incidence matrix $A_{G_2} = A_{G_2}^{\bullet 1}$. The relations between the arcs are stored in the array $\mathbf{R}_{G_2}$ (see Table 3).

We give the detailed computation of the matrix $\mathbf{A}_{G_2}^{\bar{\bullet}*}$ in the sequel:

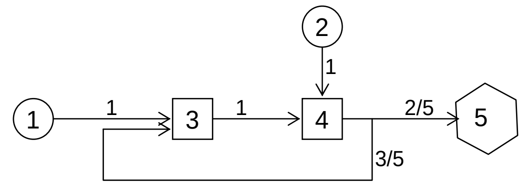

Figure 4. Tennenbaum-like net.

Table 3. Array $\mathbf{R}_{G_2}$

	[1;3]	[2;4]	[3;4]	[4;3]	[4;5]
[1;3]	id	⊥	Ø	Ø	Ø
[2;4]	⊥	id	Ø	Ø	Ø
[3;4]	Ø	Ø	id	Ø	Ø
[4;3]	Ø	Ø	Ø	id	⊥
[4;5]	Ø	Ø	Ø	⊥	id

$$\mathbf{A}_{G_2}^{\bar{\bullet}1} = \begin{pmatrix} \underline{0} & \underline{0} & [1;3] & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & [2;4] & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & [3;4] & \underline{0} \\ \underline{0} & \underline{0} & [4;3] & \underline{0} & [4;5] \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{pmatrix}$$

$$\mathbf{A}_{G_2}^{\bullet 2} = \begin{pmatrix} \underline{0} & \underline{0} & \underline{0} & [1;3][3;4] & \underline{0} \\ \underline{0} & \underline{0} & [2;4][4;3] & \underline{0} & [2;4][4;5] \\ \underline{0} & \underline{0} & [3;4][4;3] & \underline{0} & [3;4][4;5] \\ \underline{0} & \underline{0} & \underline{0} & [4;3][3;4] & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{pmatrix} = \mathbf{A}_{G_2}^{\bar{\bullet}2}$$

$$\mathbf{A}_{G_2}^{\bullet 3} = \begin{pmatrix} \underline{0} & \underline{0} & [1;3][3;4][4;3] & \underline{0} & [1;3][3;4][4;5] \\ \underline{0} & \underline{0} & \underline{0} & [2;4][4;3][3;4] & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & [4;3][3;4][4;5] \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{pmatrix}$$

and

$$\mathbf{A}_{G_2}^{\bar{\bullet}3} = \begin{pmatrix} \underline{0} & \underline{0} & [1;3][3;4][4;3] & \underline{0} & [1;3][3;4][4;5] \\ \underline{0} & \underline{0} & \underline{0} & [2;4][4;3][3;4] & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{pmatrix}$$

Let us explain why $\mathbf{A}_{G_2}^{\bar{\bullet}3} = \underline{0}$. Indeed, we have:

$$\begin{aligned} \mathbf{A}_{G_2}^{\bar{\bullet}3}(4,5) &= \mathbf{A}_{G_2}^{\bullet 2}(4,\cdot)\,\bar{\bullet}\,\mathbf{A}_{G_2}^{\bar{\bullet}1}(\cdot,5) \\ &= (\underline{0}, \underline{0}, \underline{0}, [4;3][3;4], \underline{0})\,\bar{\bullet}\,(\underline{0}, \underline{0}, \underline{0}, [4;5], \underline{0})^t \\ &= \underline{0}\,\bar{\bullet}\,\underline{0} \,\bar{\bigcup}\, \underline{0}\,\bar{\bullet}\,\underline{0} \,\bar{\bigcup}\, \underline{0}\,\bar{\bullet}\,\underline{0} \,\bar{\bigcup}\, [4;3][3;4]\,\bar{\bullet}\,[4;5] \,\bar{\bigcup}\, \underline{0}\,\bar{\bullet}\,\underline{0} \\ &= \overline{[4;3][3;4][4;5]} \\ &= \underline{0}, \text{ by definition of } \overline{\;\;}. \end{aligned}$$

We stop computation because:

$$\mathbf{A}_{G_2}^{\bar{\bullet}4} = \begin{pmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{pmatrix}$$

Finally, we have $\mathbf{A}_{G_2}^{\bar{*}} = \mathbf{I} \bar{\bigcup} \mathbf{A}_{G_2}^{\bar{\bullet}1} \bar{\bigcup} \mathbf{A}_{G_2}^{\bar{\bullet}2} \bar{\bigcup} \mathbf{A}_{G_2}^{\bar{\bullet}3}$:

$$\mathbf{A}_{G_2}^{\bar{*}} = \begin{pmatrix} \underline{1} & \underline{0} & \{[1;3],[1;3][3;4][4;3]\} & [1;3][3;4] & [1;3][3;4][4;5] \\ \underline{0} & \underline{1} & [2;4][4;3] & \{[2;4],[2;4][4;3][3;4]\} & [2;4][4;5] \\ \underline{0} & \underline{0} & \{\underline{1},[3;4][4;3]\} & [3;4] & [3;4][4;5] \\ \underline{0} & \underline{0} & [4;3] & \{\underline{1},[4;3][3;4]\} & [4;5] \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} \end{pmatrix}$$

For example, let us give the closed formula for emergy circulating on arc [4; 5]. First, look at the following column:

$$\begin{pmatrix} \mathbf{A}_{G_2}^{\bar{*}}(1,5) \\ \mathbf{A}_{G_2}^{\bar{*}}(1,5) \end{pmatrix}$$

and take all words which terminate by [4; 5], that is: $\varepsilon([4;5]) = \{[1;3][3;4][4;5], [2;4][4;5]\}$ and compute $\varphi(\varepsilon([4;5]))$ as follows:

1. 1 and 2 are emergy sources, thus by definition of $\perp$ we have $[1;3] \perp [2;4]$ (note that this was already summarized in the array $\mathbf{R}_{G_2}$.
2. Rewrite $\varphi(\varepsilon([4;5]))$ as:

$$\varphi(\varepsilon([4;5])) = \varphi([1;3]\{[3;4][4;5]\} \bigcup [2;4]\{[4;5]\}),$$

with $[1;3] \perp [2;4]$.

3. Apply (φ.4.1) with $m = \underline{1}$, $k = 2$, $a_1 = [1;3]$, $\mathbb{U}_1 = \{[3;4][4;5]\}$, $a_2 = [2;4]$ and $\mathbb{U}_2 = \{[4;5]\}$. Then,

$$\varphi(\varepsilon([4;5])) = \varphi([1;3]\{[3;4][4;5]\}) \otimes \varphi([2;4]\{[4;5]\})$$

4. Compute $\varphi([2;4]\{[4;5]\})$ as follows:

$\varphi([2;4]\{[4;5]\})$
$= \mathrm{POW}(\varphi([2;4]),\varphi(\{[4;5]\}))$
by (φ.3, $m = [2;4]$ $\mathbb{U} = \{[4;5]\}$)
$= \mathrm{POW}(\mathrm{POW}(\theta(2),\omega([2;4]),\varphi(\{[4;5]\})))$
by (φ.2, $l = 2$, $l' = 4$)
$= \mathrm{POW}(\mathrm{POW}(\theta(2),\omega([2;4]),\varphi([4;5])))$
by (φ.1, noticing m = [4; 5])
$= \mathrm{POW}(\mathrm{POW}(\theta(2),\omega([2;4])),\omega([4;5]))$
by φ.2, noticing that $4, 5 \notin \mathbb{L}_2^s$

5. Compute $\varphi([1;3]\{[3;4][4;5]\}$ as follows:

$\varphi([1;3]\{[3;4][4;5]\})$
$= \mathrm{POW}(\varphi([1;3]),\varphi(\{[3;4][4;5]\}))$
$= \mathrm{POW}(\varphi([1;3]),\varphi([3;4]\{[4;5]\}))$
by definition of the Minkowski product:
$= \mathrm{POW}(\varphi([1;3]),\mathrm{POW}(\varphi([3;4]\{[4;5]\})))$
by (φ.3) and $m = [3;4]$ $\mathbb{U} = \{[4;5]\}$

Then, applying (φ.2) to φ([1;3]), φ([3; 4]) and φ([4; 5]), we have:

$\varphi([1;3]) = \mathrm{POW}(\theta(1),\omega([1;3]))$

$\varphi([3;4]) = \omega([3;4])$

$\varphi([4;5]) = \omega([4;5])$.

Finally, by associativity of POW we have:

$$\varphi(\varepsilon([4;5])) = \mathrm{POW}(\theta(1),\omega([1;3]),\omega([3;4]),\omega([4;5])) \otimes \mathrm{POW}(\theta(2), \omega([2;4]),\omega([4;5]))$$

Using usual notations we have:

$$\varphi(\varepsilon([4;5])) = \theta(1)\,\omega([1;3])\,\omega([3;4])\,\omega([4;5]) + \theta(2)\,\omega([2;4])\,\omega([4;5])$$

Numerical application: Brown and Herendeen (1996) chose for emergy sources $\theta(1) = 400$, $\theta(2) = 100$ and for weights of the arcs $\omega([4;5]) = 2/5$, $\omega([4;3]) = 3/5$, $\forall\, a \in \mathbb{A}_2 \setminus \{[4;3], [4;5]\}$ $\omega(a)=1$. Thus, one gets:

$$\varphi(\varepsilon([4;5])) = 400\;0.4 + 100\;0.4 = 200,$$

which is the value obtained at the output of the graph (Brown and Herendeen 1996 (Figure 8b, p. 226)).

Remark 5.1. The emergy computed corresponds to the entry (4, 5) of the matrix FRM in the Tennenbaum's program (see Tennenbaum (pp. 122 ~ 126) (1988)).

5.2. Example 3.1 Continued

Let us remark that the emergy graph of this example (see Figure 3) does not have splits. It possesses only co-products at nodes 3 and 6. From the computation of the matrix $\mathbf{A}_{G_1}^{\bar{*}}$, we have:

$$\mathbf{A}_{G_1}^{\bar{*}}(1,\cdot) = \begin{pmatrix} \underline{0} \\ \underline{0} \\ [1;3] \\ [1;3][3;4] \\ \{[1;3][3;5],[1;3][3;5][5;6][6;5],[1;3][3;4][4;6][6;5]\} \\ \{[1;3][3;4][4;6],[1;3][3;5][5;6],[1;3][3;4][4;6][6;5][5;6]\} \\ \{[1;3][3;4][4;6][6;7],[1;3][3;5][5;6][6;7]\} \end{pmatrix}^t$$

and

$$\mathbf{A}_{G_1}^{\bar{*}}(2,\cdot) = \begin{pmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ [2;4] \\ [2;4][4;6][6;5] \\ \{[2;4][4;6],[2;4][4;6][6;5][5;6]\} \\ [2;4][4;6][6;7] \end{pmatrix}^t$$

As an illustrative example, let us compute the emergy flowing on the arc [6; 5], i.e. $\varphi(\varepsilon([6;5]))$ with ε([6; 5]) = {[1; 3] [3; 5] [5; 6] [6; 5], [1; 3][3; 4] [4; 6] [6; 5], [2; 4][4; 6][6; 5]}.

Because $1, 2 \in \mathbb{L}_1^s$ we have: [1; 3] ⊥ [2; 4], by definition of ⊥. Thus, we express ε([6; 5]) as follows:

ε([6; 5]) = [1; 3] $\mathbb{U}_1 \bigcup$ [2; 4] $\mathbb{U}_2$,

with $\mathbb{U}_1$ = {[3; 5] [5; 6] [6; 5], [3; 4] [4; 6] [6; 5]} and $\mathbb{U}_2$ = {[4; 6] [6; 5]}. And we obtain:

φ (ε([6; 5])) = φ ([1; 3] $\mathbb{U}_1$ ∪ [2; 4] $\mathbb{U}_2$) = φ ([1; 3] $\mathbb{U}_1$) ⊗ φ([2; 4] $\mathbb{U}_2$) by (φ.4.1)

By an easy computation we have:

φ ([2; 4] $\mathbb{U}_2$)=POW(θ(2), ω([2; 4] [4; 6][6; 5])),

with ω([2; 4] [4; 6] [6; 5]) = POW(ω([2; 4]), ω([4; 6]), ω([6; 5])). Let us detail the computation of φ([1; 3] $\mathbb{U}_1$). It comes:

φ ([1; 3] $\mathbb{U}_1$) = POW(φ ([1; 3]),φ ($\mathbb{U}_1$)) by (φ.3)
= POW(POW(θ(1),ω ([1;3])), φ($\mathbb{U}_1$)) by (φ.2)

Now, we just have to compute φ($\mathbb{U}_1$). We remark that: $\mathbb{U}_1$ = [3; 4] {[4; 6][6; 5]} ∪ [3; 5] {[5; 6] [6; 5]}, with [3; 4] // [3; 5] because there is a co-product at node 3. Then, by applying (φ.4.2) we have: φ($\mathbb{U}_1$) = φ([3; 4]{[4; 6][6; 5]}) ⊕ φ([3; 5] {[5; 6] [6; 5]}).

Using (φ.3), (φ.2), associativity of POW and (ω.2) we have: φ([3; 4] {[4; 6] [6; 5]}) = ω([3; 4] [4; 6] [6; 5]) and φ([3; 5] {[5; 6] [6; 5]}) = ω([3; 5] [5; 6] [6; 5]). with ω([3; 4] [4; 6] [6; 5]) = POW(ω([3; 4]), ω([4; 6]), ω([6; 5])) and ω([3; 5] [5; 6] [6; 5]) = POW(ω([3; 5]), ω([5; 6]), ω([6; 5])). Finally, we obtain: φ(ε([6; 5])) = POW(θ(2), ω([2; 4] [4; 6] [6; 5])) ⊗ (POW (θ(1), ω([3; 4] [4; 6] [6; 5])) ⊕ POW(θ(1), ω([3; 5][5; 6][6;5]))). Using usual notations we have:

$\varphi(\varepsilon([6; 5])) = \theta(2)\omega([2; 4])\omega([4; 6])\omega([6; 5])) + \max(\theta(1)\omega([3; 4])\omega([4; 6])\omega([6; 5]), \theta(1)\omega([3; 5])\omega([5; 6])\omega([6; 5]))$.

Numerical application: Brown and Herendeen (1996, Figure 9b) chose for emergy sources θ(1) = 400, θ(2) = 100 and for the weights, $\forall\ a \in \mathbb{A}_1$ ω(a) = 1. Thus, one gets:

φ (ε([6; 5])) = 100 + max(400, 400) = 500.

5.3. Emergy Graph with Splits and one Co-product

Let us consider the emergy graph of Figure 5 as illustrated by Li et al. [Figure 8 and 9] (2010). There are splits at nodes 3, 5, 6, 7 and 10, and a co-product at node 4. The set of sources is $\mathbb{L}^s$ = {1, 2}, the set of internal nodes is $\mathbb{L}^i$ = {3, 4, 5, 6, 7, 8, 9, 10} and the set of the output nodes is $\mathbb{L}^o$ = {11, 12, 13, 14}. Because 1 and 2 are sources we have: [1; 3] ⊥ [2; 10]. Because 3, 5, 6, 7 and 10 are spited we have: [3; 4] ⊥ [3; 5], [6; 8] ⊥ [6; 9], [7; 9] ⊥ [7; 10] and [10; 4] ⊥ [10; 11]. Because of the co-product at node 4 we have: [4; 6] // [4; 7].

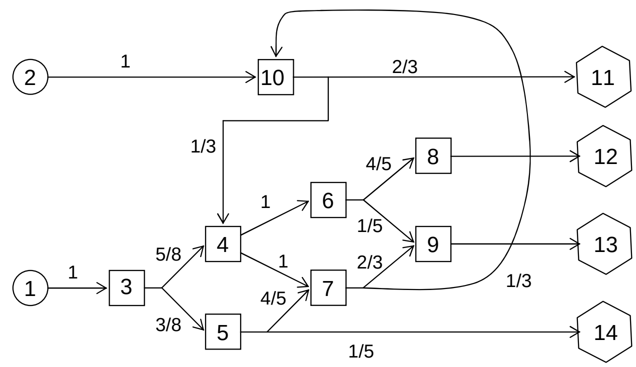

Figure 5. Net with splits and one co-product at node 4.

Let us give the main steps of the computation of the emergy circulating on the arc [9; 13] denoted φ(ε([9; 13])), recalling that ε([9; 13]) is the set of all physical emergy paths ending by [9; 13].

Step 1. The computation of the set ε([9; 13]) based on the computation of the *-closure of the incidence matrix of the graph gives:

$$\varepsilon\ ([9; 13]) = \begin{Bmatrix} [1;3][3;5][5;7][7;10][10;4][4;6][6;9][9;13], \\ [1;3][3;5][5;7][7;9][9;13], \\ [1;3][3;4][4;6][6;9][9;13], [1;3][3;4][4;7][7;9][9;13], \\ [2;10][10;4][4;6][6;9][9;13], [2;10][10;4][4;7][7;9][9;13] \end{Bmatrix}$$

Step 2. We apply (ω.0) ~ (ω.2), (φ.0) ~ (φ.4) and obtain the following closed formula for φ(ε([9; 13])):

φ (ε([9; 3])) = POW(θ(1), ω ([1; 3] [3; 5] [5; 7] [7; 10][10; 4][4; 6][6; 9] [9; 13])) ⊗ POW(θ(1), ω([1; 3][3; 5][5; 7][7; 9][9; 13]))⊗(POW(θ(1), ω ([1; 3][3; 4][4; 6][6; 9][9; 13])) ⊕ POW (θ(1), ω ([1;3][3;4][4;7][7;9][9;13]))) ⊗ (POW(θ(2), ω ([2; 10] [10; 4][4; 6][6; 9][9; 13])) ⊕ POW(θ (2), ω ([2; 10][10; 4][4; 7] [7; 9][9; 13])))

Using usual notations we have: φ(ε([9;13])) = θ(1) ω([1;

3][3; 5][5; 7][7; 10][10; 4][4; 6][6; 9][9; 13]) + θ(1) ω([1; 3][3; 5][5; 7][7; 9][9; 13]) + max(θ(1) ω([1; 3][3; 4][4; 6][6; 9][9; 13]), θ(1) ω([1; 3] [3; 4] [4; 7] [7; 9] [9; 13])) + max (θ(2) ω([2; 10] [10; 4] [4; 6] [6; 9] [9; 13]), θ(2) ω([2; 10] [10; 4] [4; 7] [7; 9] [9; 13])).

Numerical application. Li et al. (2010) chose for emergy sources θ(1) = 1000, θ(2) = 500 and for the weights ω([1; 3]) = ω([2; 10]) = ω([4; 6]) = ω([4; 7]) = ω([9; 13])=1 and ω([3; 4]) = 5/8, ω([3; 5]) = 3/8, ω([5; 7]) = 4/5, ([6; 9]) = 1/5, ω([7; 9]) = 2/3, ω([7; 10]) = ω([10; 4]) = 1/3. The numerical application gives:

$$\varphi(\varepsilon([9;\,13])) = 1000\ \frac{3}{8}\frac{4}{5}\frac{1}{3}\frac{1}{3}\frac{1}{5} + 1000\ \frac{3}{8}\frac{4}{5}\frac{2}{3} + \max(1000\ \frac{1}{3}\frac{1}{5},\ 1000\ \frac{1}{3}\frac{2}{3}) + \max(500\ \frac{1}{3}\frac{1}{5},\ 500\ \frac{1}{3}\frac{2}{3}) = \frac{20}{3} + 200 + \max(125, \frac{1250}{3}) + \max(\frac{100}{3}, \frac{1000}{9}) = \frac{6610}{9}(\approx 734.44)$$

6. Discussion

Let us first remark that the method/algorithm has two main parts. The first part is the enumeration of emergy flows. The second part is the numerical computation of the emergy flowing on an arc of the emergy graph. The emergy rules R1 ~ R4 do not clearly reveal this difference. Nevertheless,

- The rule R1 has been translated as a particular case of axiom (φ.4.1) with ∀ i = 1,… ,k: $a_i \in [\mathbb{L}^s; l]$, $\mathbb{U}_i = \{[l; l']\}$ for some l, $l' \in \mathbb{L}\backslash\mathbb{L}^s$, and the axioms (φ.0 ~ φ.3). This rule is illustrated in e.g. Brown and Herendeen (Figure 6b, p. 225) (1996). However, let us remark that this rule is not always written the same way in the literature (see e.g. Sciubbia and Ulgiati (2005) --also used in the Introduction of this paper--, Li et al. (2010), Lazzaretto (2009), Ridolfi and Bastianoni (2008)). The case of one output is completely treated by axioms (φ.4.0) and (φ.0 ~ φ.3).
- The rule R2 concerning splits has been translated by axioms (ω.1), (φ.4.1) and (φ.0 ~ φ.3).
- The rule R3 is translated as a particular case of the axiom (φ.4.2) with $\mathbb{U}_i = \{\underline{1}\}$, i = 1, …, k and the application of (φ.0 ~ φ.3).
- The rule R4 concerning the double counting problem is solved as follows. For the qualitative part we have introduced the idempotent semiring $\mathbb{F}$ defined by (5). Then, we have defined the relations (or Thue congruences) $\overset{*}{\Leftrightarrow}_1$ and $\overset{*}{\Leftrightarrow}_2$ and the quotient relation $\overset{*}{\Leftrightarrow}_1 / \overset{*}{\Leftrightarrow}_2$. This quotient relation allows us to define the operator $\overline{\ }$ and the operation $\overline{\cup}$ and $\bullet$. The Theorem 3.1 and the step B of the Algorithm Section 4 clearly show that the qualitative double counting problem of emergy is solved in an algebraic structure of the form $(\mathrm{Mat}_n(\overline{\mathbb{F}^*_0}), \overline{\cup}, \bullet, (\underline{0}), \mathbf{I})$.
- The rule R4.2 (i.e. the quantitative part of R4) is traduced by the application of the axioms (φ.4.2) and (φ.0 ~ φ.3).

The axioms (p_0 ~ p_6) characterize the binary relations Ø, id, ⊥ and //. They are used to insure the coherency of the array $\mathbf{R}_G$ defined by (20) and to prove that the computation algorithm of the emergy on an arc terminates.

7. Conclusions

In this paper we have extended the Tennenbaum's Track Summing method to the case of emergy networks with both splits and co-products. To obtain this extension we have reformulated the emergy rules R1 ~ R4 (see the Introduction) into the axiomatic basis (p_0 ~ p_6) (see subsection 3.1), and axioms (ω.0 ~ ω.2), (φ.0 ~ φ.4) (see subsection 3.3). The main concepts used are idempotent semirings instead of linear algebra and the tree property allowing a recursive definition of the emergy flowing on an arc of the emergy graph. The method is programmable and partially parallelizable. Even if authors cannot formally prove that the axiomatic basis is logically equivalent to the rules R1 ~ R4 the method has been tested on classical examples of the literature and has given the same results. A last point one should note is that the method is not only a computational method. It also provides a rigorous framework based on an axiomatic basis to complete the emergy evaluation of an emergy graph.

Appendix A: Explanation of axiom (φ.4.1)

Let us consider the emergy graph of Figure 6 such that $\theta(1)$ = 300. Let us compute φ(ε([6; 5])). We have ε([6; 5]) = {[1; 3][3; 4][4; 6][6; 5], [1; 3][3; 5][5; 6][6; 5]}. Because there is a split at node 3: [3; 4] ⊥ [3; 5], thus the set ε([6; 5]) is decomposed as follows:

ε ([6; 5]) = [1; 3][3; 4] $\mathbb{U}_1$ ∪ [1; 3][3; 5] $\mathbb{U}_2$

with: $\mathbb{U}_1$ = {[4; 6][6; 5]} and $\mathbb{U}_2$ = {[5; 6][6; 5]}.

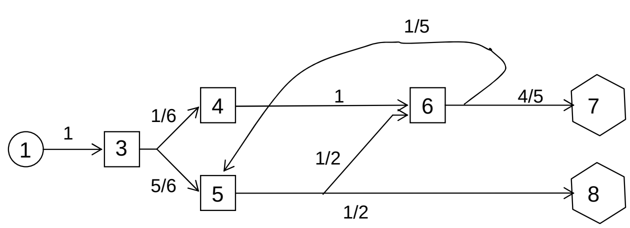

Figure 6. Emergy graph with split.

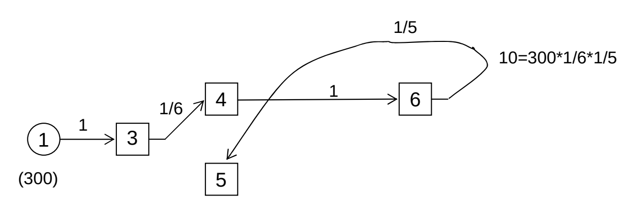

Figure 7. First pathway from 1 to 5.

Figure 7 explains how to compute the emergy flowing on arc [6; 5] of the system where there is only the arc [3; 4] after the upstream flow [1; 3], that is $\varphi([1; 3][3; 4]\ \mathbb{U}_1)$.

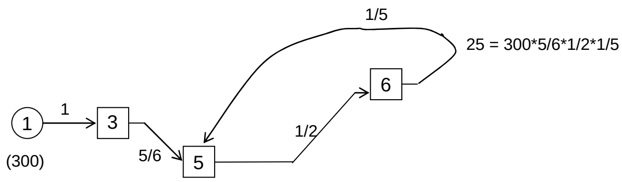

Figure 8. Second pathway from 1 to 5.

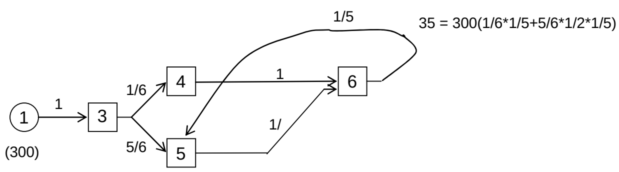

Figure 9. Total emergy flowing on arc [6;5].

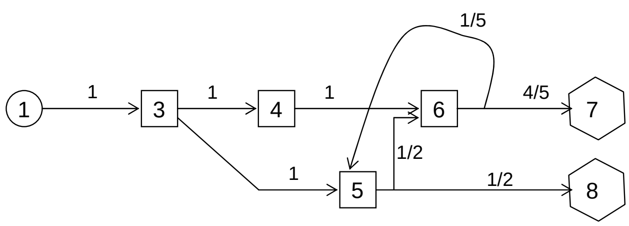

Figure 10. Emergy graph with co-product.

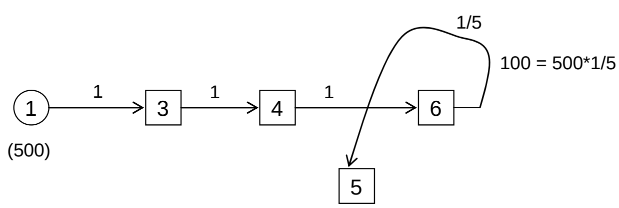

Figure 11. Emergy on the first pathway from 1 to 5.

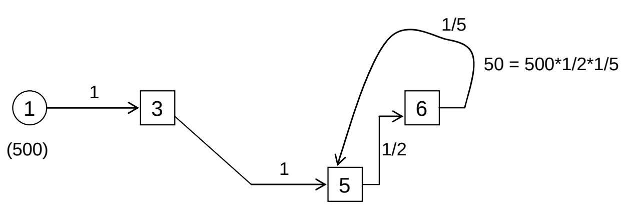

Figure 12. Emergy on the second pathway from 1 to 5.

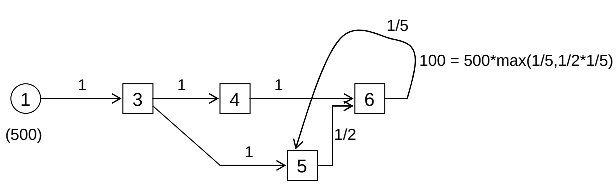

Figure 13. Total emergy flowing on arc [6;5].

Figure 8 explains how to compute the emergy flowing on arc [6; 5] of the system where there is only the arc [3; 5] after the upstream flow [1; 3], that is $\varphi([1;3][3;5]\ \mathbb{U}_2)$.

Finally, when reunited Figure 9 explains how to compute the whole emergy flowing on arc [6; 5] and illustrates the formula:

$$\varphi([1; 3]\,[3; 4]\ \mathbb{U}_1 \cup [1; 3]\,[3; 5]\ \mathbb{U}_2) = \varphi([1; 3]\,[3; 4]\ \mathbb{U}_1) \otimes \varphi([1; 3]\,[3; 5]\ \mathbb{U}_2).$$

In the general case we have:

$$\varphi(\bigcup_{i=1}^{k} m\ a_i\ \mathbb{U}_i) = \bigotimes_{i=1}^{k} \varphi(m\ a_i\ \mathbb{U}_i),$$

and the operation $\otimes$ (i.e. addition) is well associated with relation $\perp$.

Appendix B: Explanation of axiom (φ.4.2)

Let us consider the emergy graph of Figure 10 such that $\theta(1) = 500$. Let us compute $\varphi(\varepsilon([6; 5]))$. We have $\mathcal{E}([6; 5]) = \{[1; 3]\,[3; 4]\,[4; 6][6; 5], [1; 3]\,[3; 5]\,[5; 6]\,[6; 5]\}$. Because there is a co-product at node 3: [3; 4] // [3; 5], thus the set $\mathcal{E}([6; 5])$ is decomposed as follows:

$$\mathcal{E}([6; 5]) = [1; 3]\,[3; 4]\ \mathbb{U}_1 \cup [1; 3]\,[3; 5]\ \mathbb{U}_2$$

with: $\mathbb{U}_1 = \{[4; 6]\,[6; 5]\}$ and $\mathbb{U}_2 = \{[5; 6][6; 5]\}$.

Figure 11 explains how to compute the emergy flowing on arc [6; 5] of the system where there is only the arc [3; 4] after the upstream flow [1; 3], that is $\varphi([1; 3][3; 4]\ \mathbb{U}_1)$.

Figure 12 explains how to compute the emergy flowing on arc [6; 5] of the system where there is only the arc [3; 5] after the upstream flow [1; 3], that is $\varphi([1; 3]\,[3; 5]\ \mathbb{U}_2)$.

Finally, when reunited Figure 13 explains how to compute the whole emergy flowing on arc [6; 5] and illustrates the formula:

$$\varphi([1; 3]\,[3; 4]\ \mathbb{U}_1 \cup [1; 3]\,[3; 5]\ \mathbb{U}_2) = \varphi([1; 3]\,[3; 4]\ \mathbb{U}_1) \oplus \varphi([1; 3]\,[3; 5]\ \mathbb{U}_2).$$

In the general case we have:

$$\varphi(\bigcup_{i=1}^{k} m\ a_i\ \mathbb{U}_i) = \bigoplus_{i=1}^{k} \varphi(m\ a_i\ \mathbb{U}_i),$$

and the operation $\oplus$ (i.e. maximum) is well associated with the co-product //.

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