Journal of Environmental Informatics 17(2) 54-64 (2011)

Journal of Environmental Informatics

www.iseis.org/jei

Analysis of Solution Methods for Interval Linear Programming

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Received 15 August 2009; revised 25 January 2010; accepted 29 Apirl 2010; published online 10 June 2011

ABSTRACT. In this paper, solution methods for ILP are studied. First of all, the principals and assumptions of two-step method (TSM) are analyzed. Secondly, the definition of feasible decision space for ILP is introduced. Also the existence of infeasible solutions and how these solutions are generated in TSM is examined. Thirdly, new solution method named three-step method (ThSM) is developed for solving ILP models. It is based on three proposed steps: TSM, feasibility test, and constricting method. The main advantage of ThSM is that no infeasible solutions would be included in the obtained results. Moreover, ThSM can generated interval solutions and does not have high computational requirements. An example has been presented to explain in detail the solution process of ThSM. Fourthly, three scenarios of Monte Carlo simulations have been introduced to further explore the detailed solutions for ILP. The results demonstrate that when all coefficients of ILP are assumed to obey normal or uniform distribution the developed methods are applicable. Under other distribution assumptions for coefficients in ILP, further studies should be developed.

Keywords: interval linear programming, approximate method, two-step method, three-step method, feasibility test, constricting method, Monte Carlo simulation

1. Introduction

ILP is an effective tool for supporting decisions under uncertainty. Among methods for solving ILP problem, Huang (1994) proposed a two-step approach which was extensively used by many researchers (Nie et al., 2007; Maqsood et al., 2005; Liu et al., 2009; Lv et al., 2010; Sun and Huang, 2010; Yan et al., 2010; Cao et al. 2011). The detailed algorithm and a demonstrating example were published in Huang et al. (1992, 1995). In this study, an ILP model will be considered as follows:

$$\text{Min } f^{\pm} = C^{\pm}X^{\pm} \tag{1a}$$

Subject to

$$A^{\pm}X^{\pm} \le B^{\pm} \tag{1b}$$

$$X^{\pm} \ge 0 \tag{1c}$$

where

$$C^{\pm} = [c_1^{\pm}, ..., c_n^{\pm}]\,,\ C^{-} = [c_1^{-}, ..., c_n^{-}]\,,\ C^{+} = [c_1^{+}, ..., c_n^{+}]\,;$$

$$A^{\pm} = \begin{bmatrix} a_{11}^{\pm}, ..., a_{1n}^{\pm} \\ ... \\ a_{m1}^{\pm}, ..., a_{mn}^{\pm} \end{bmatrix},\ A^{-} = \begin{bmatrix} a_{11}^{-}, ..., a_{1n}^{-} \\ ... \\ a_{m1}^{-}, ..., a_{mn}^{-} \end{bmatrix},\ A^{+} = \begin{bmatrix} a_{11}^{+}, ..., a_{1n}^{+} \\ ... \\ a_{m1}^{+}, ..., a_{mn}^{+} \end{bmatrix};$$

$$B^{\pm} = \begin{bmatrix} b_1^{\pm} \\ ... \\ b_m^{\pm} \end{bmatrix},\ B^{-} = \begin{bmatrix} b_1^{-} \\ ... \\ b_m^{-} \end{bmatrix},\ B^{+} = \begin{bmatrix} b_1^{+} \\ ... \\ b_m^{+} \end{bmatrix};$$

where $c_j^{\pm}, a_{ij}^{\pm}, b_i^{\pm} \in R^{\pm}$, and $R^{\pm}$ denotes a set of interval numbers. The lower and upper bounds of the intervals are assumed to hold the same sign. For example, an interval like [-2, 1] will not be considered in this study.

To solve model (1), Huang (1992) developed an interactive approach named two-step method (TSM). The ILP model was transformed to two submodels with deterministic coefficients. For the convenience of further analyzing the TSM algorithm, we present the two submodels. The submodel corresponding to f^{-} should be formulated first (assume $b_i^{\pm} > 0$ and $f^{\pm} > 0$):

$$\text{Min } f^{-} = \sum_{j=1}^{k} c_j^{-} x_j^{-} + \sum_{j=k+1}^{n} c_j^{-} x_j^{+}, \tag{2a}$$

Subject to:

$$\sum_{j=1}^{k} \left|a_{ij}^{\pm}\right|^{+} Sign(a_{ij}^{\pm}) x_j^{-} + \sum_{j=k+1}^{n} \left|a_{ij}^{\pm}\right|^{-} Sign(a_{ij}^{\pm}) x_j^{+} \le b_i^{+} \quad i = 1, 2, ..., m \tag{2b}$$

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ISSN: 1726-2135 print/1684-8799 online

$$x_j^- \geq 0, j = 1, 2, ..., k \tag{2c}$$

$$x_j^+ \geq 0, \ j = k+1, k+2, ..., n \tag{2d}$$

Solutions of x_{jopt}^- ($j = 1, 2, ..., k$) and x_{jopt}^+ ($j = k$+1, k+2, …, n) can be obtained through solving submodel (2). Then the submodel corresponding to f^+ can be formulated as follows (assume that $b_i^\pm > 0$ and $f^\pm > 0$):

$$\text{Min} \ \ f^+ = \sum_{j=1}^{k} c_j^+ x_j^+ + \sum_{j=k+1}^{n} c_j^+ x_j^- \tag{3a}$$

Subject to:

$$\sum_{j=1}^{k} \left| a_{ij}^\pm \right|^- Sign(a_{ij}^\pm) x_j^+ + \sum_{j=k+1}^{n} \left| a_{ij}^\pm \right|^+ Sign(a_{ij}^\pm) x_j^- \leq b_i^- \ \ i = 1, 2, ..., m \tag{3b}$$

$$x_j^+ \geq x_{j\,opt}^-, \ j = 1, \ 2, \ ..., \ k \tag{3c}$$

$$0 \leq x_j^- \leq x_{j\,opt}^+, \ j = k+1, \ k+2, \ ..., \ n \tag{3d}$$

Hence, solutions of x_{jopt}^+ ($j = 1, 2, \ldots, k$) and x_{jopt}^- ($j = k+1, k+2, \ldots, n$) can be obtained through solving submodel (3). Thus, the final solution of $f_{opt}^\pm = [f_{opt}^-, f_{opt}^+]$ and $x_{jopt}^\pm = [x_{jopt}^-, x_{jopt}^+]$ can be obtained.

TSM has been applied for solving a number of ILP problems (Huang 1998; Huang and Loucks, 2000; Huang et al., 2001; Maqsood et al., 2005; Cheng et al., 2009; Cao et al., 2010a, b; Gao et al., 2010; He et al., 2010). The application fields include municipal solid waste management, water resources allocation, air quality control planning, and energy systems planning. TSM was widely used due to its following advantages: (1) TSM did not lead to high computational requirement; (2) the interval solutions could help generate a series of decision alternatives. Thus, solutions of TSM were adjustable and were effective in reflecting complexities in real-world decision problems. However, results obtained through TSM might contain solutions which violated the constraints. This disadvantage might result in significant system-failure risk.

The objective of this study is to develop an improved solution method for the ILP problems so that the constraints will no longer be violated. In the second section, the principals and assumptions for formulating the two submodels in TSM will be analyzed. The definition of solving violation and how such violations are generated will be discussed. Consequently, an improved approach, named three-step method (ThSM) will be developed. Solutions for ThSM will not violate the constraints. An example will be presented to explain in detail the process of ThSM . In the discussion section, three scenarios of Monte Carlo simulations will be introduced to further explore and demonstrate the solutions.

2. Analysis of Two-Step Method (TSM)

2.1. Two Submodels in TSM

2.1.1. The Meaning of Combinations f and b_i

When TSM is adopted to solve an ILP problem, it implies that the decision makers are optimistic about the studied case. The two submodels in TSM are not set in parallel since solutions of the first submodel should be incorporated within the second submodel as additional constraints. In other words, the first submodel holds priority. Take the case of MSW management as an example (Huang, 1994), the objective is to minimize the total cost of waste transportation and disposal. When TSM is used to solve the formulated ILP model, the submodel corresponding to the lower bound of the total cost should be solved firstly. Thus the solutions will be used as new constraints in the submodel corresponding to the upper bound of the total cost. In this sense, the lower bound of system cost is considered more important than the upper one.

Moreover, suppose landfill is one of the waste disposal facilities. Its upper bound capacity is adopted in the first submodel, implying a larger decision space. This means that decision makers are more interested in the solutions corresponding to the lower-bound cost. These demonstrate that TSM provided relatively optimistic solutions for ILP problem.

2.1.2. Analysis of Combinations $a_{ij}x_j$

Interactivities exist between coefficients (a_{ij}) and decision variables (x_j) in the left-hand sides of the constraints. For both submodels in TSM, we have the following combinations of them:

$$\left| a_{ij}^\pm \right|^- Sign(a_{ij}^\pm) x_j^+ = \begin{cases} a_{ij}^- x_j^+, \ a_{ij}^\pm \geq 0 \\ a_{ij}^+ x_j^+, \ a_{ij}^\pm < 0 \end{cases} \tag{4a}$$

$$\left| a_{ij}^\pm \right|^+ Sign(a_{ij}^\pm) x_j^- = \begin{cases} a_{ij}^+ x_j^-, \ a_{ij}^\pm \geq 0 \\ a_{ij}^- x_j^-, \ a_{ij}^\pm < 0 \end{cases} \tag{4b}$$

Combinations $a_{ij}^- x_j^+$ and $a_{ij}^+ x_j^-$ are used when $a_{ij}^\pm \geq 0$, and $a_{ij}^+ x_j^+$ and $a_{ij}^- x_j^-$ are used when $a_{ij}^\pm < 0$. When $a_{ij}^\pm \geq 0$, their relations should be $a_{ij}^- x_j^- \leq a_{ij}^- x_j^+$, $a_{ij}^+ x_j^- \leq a_{ij}^+ x_j^+$. Combinations $a_{ij}^- x_j^+$ and $a_{ij}^+ x_j^-$ are selected due to the assumption that $a_{ij}^\pm x_j^\pm$ takes value randomly, and holds the following characteristics:

$P_r\{ax = y_1\} \leq P_r\{ax = y_2\}$, where $a \in a_{ij}^\pm$, $x \in x_j^\pm$ and $a_{ij}^- x_j^- \leq y_1 \leq y_2 \leq 0.5 a_j^- x_j^- + 0.5 a_j^+ x_j^+$;

$P_r\{ax = y_1\} \geq P_r\{ax = y_2\}$,where $a \in a_{ij}^\pm$, $x \in x_j^\pm$ and $0.5 a_j^- x_j^- + 0.5 a_j^+ x_j^+ \leq y_1 \leq y_2 \leq a_{ij}^+ x_j^+$.

A number of probability density functions (e.g. normal distribution) hold the above characteristic. Figure 1 shows several examples of distribution types. Therefore, we have:

$$P_r\{ax = a_j^- x_j^+\} \geq P_r\{ax = a_j^- x_j^-\}, P_r\{ax = a_j^+ x_j^+\} \tag{5a}$$

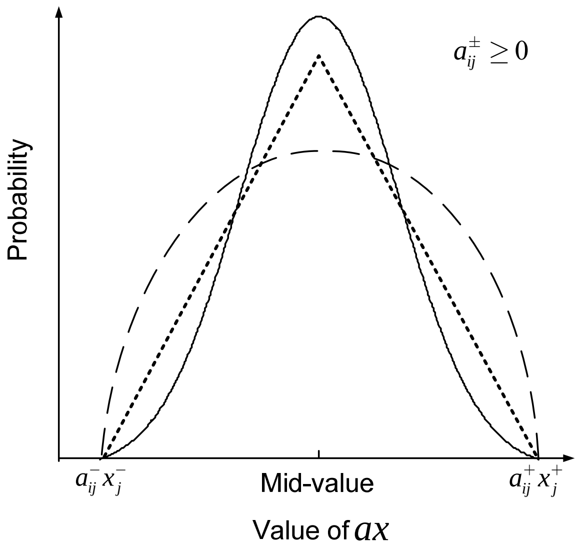

Figure 1. Distribution information of *ax*.

$$P_r\{ax = a_j^+ x_j^-\} \ge P_r\{ax = a_j^- x_j^-\},\ P_r\{ax = a_j^+ x_j^+\} \tag{5b}$$

The probability for *ax* to take values of $a_{ij}^- x_j^+$ and $a_{ij}^+ x_j^-$ is larger than that to take values of $a_{ij}^+ x_j^+$ and $a_{ij}^- x_j^-$. In this sense, it is more significant to take values of $a_{ij}^- x_j^+$ and $a_{ij}^+ x_j^-$ to represent *ax*.

On the other hand, when $a_{ij}^{\pm} < 0$, relations among all possible combinations for lower and upper bounds of $a_{ij}^{\pm}$ and $x_j^{\pm}$ should be $a_{ij}^- x_j^+ \le a_{ij}^- x_j^-$, $a_{ij}^+ x_j^+ \le a_{ij}^+ x_j^-$. Based on the same assumption of distribution type as considered in the case of $a_{ij}^{\pm} \ge 0$, we have:

$$P_r\{ax = a_j^- x_j^-\} \ge P_r\{ax = a_j^- x_j^+\},\ P_r\{ax = a_j^+ x_j^-\} \tag{6a}$$

$$P_r\{ax = a_j^+ x_j^+\} \ge P_r\{ax = a_j^- x_j^+\},\ P_r\{ax = a_j^+ x_j^-\} \tag{6b}$$

where $a \in a_{ij}^{\pm}$, $x \in x_j^{\pm}$. Therefore, the combinations of $a_{ij}^+ x_j^+$ and $a_{ij}^- x_j^-$ are used to represent *ax* when $a_{ij}^{\pm} < 0$.

In general, the setting of relations between *f* and b_i in TSM can be considered as an aggressive scenario when the decision makers are optimistic of the study problems. Other scenarios under different combinations for the bounds of *f* and b_i can also be considered. In comparison, the combinations for the bounds of a_{ij} and x_j are unique while the bases for the combination are assumptions (5a) to (6b). In other words, if $a_{ij}\, x_j$ did not follow inequalities (5a) to (6b), the interrelations between a_{ij} and x_j in TSM would become meaningless.

2.2. Feasible Decision Space for ILP

Several definitions of feasible decision space for ILP have been developed based on analysis of the uncertainties (Nakahara et al., 1992; Tong 1994). In this study, the feasible decision space for ILP model (1) is defined as follows:

$$Q = \left\{ X \,\middle|\, A^- X \le B^+,\ X \in R^n,\ X \ge 0 \right\} \tag{7}$$

This implies that, once the solutions of $X^{\pm}$ are given, then for any arbitrary value in $X^{\pm}$, all constraints can be tenable by means of adjusting the values of coefficients *A* and *B* within the ranges of $A^{\pm}$ and $B^{\pm}$.

Remark 1. Let the feasible decision spaces of models (2) and (3) be Q_1 and Q_2. Then we have: $Q_1 \subseteq Q$ and $Q_2 \subseteq Q$. For $\forall\ X^0 = [x_1^0, x_2^0, ..., x_n^0] \in Q_1$, $\forall a_{ij}^{\pm}$, *i* = 1, 2, …, *m*; *j* = 1, 2, …, *n*. Thus we have $a_{ij}^- \le |a_{ij}^{\pm}|^- Sign(a_{ij}^{\pm}) \le a_{ij}^+$, and $a_{ij}^- \le |a_{ij}^{\pm}|^+ Sign(a_{ij}^{\pm}) \le a_{ij}^+$. Since $X^0 \ge 0$, we have $a_{ij}^- x_j^0 \le |a_{ij}^{\pm}|^- Sign(a_{ij}^{\pm}) x_j^0 \le a_{ij}^+ x_j^0$, and $a_{ij}^- x_j^0 \le |a_{ij}^{\pm}|^+ Sign(a_{ij}^{\pm}) x_j^0 \le a_{ij}^+ x_j^0$. Thus $A^- X^0 = a_{i1}^- x_1^0 + ... + a_{in}^- x_n^0 \le \sum_{j=1}^{k} |a_{ij}^{\pm}|^+ Sign(a_{ij}^{\pm}) x_j^0 + \sum_{j=k+1}^{n} |a_{ij}^{\pm}|^- Sign(a_{ij}^{\pm}) x_j^0 \le b_i^+$, *i* = 1, 2, …, *m*. This means $A^- X^0 \le B^+$. Therefore we have $X^0 \in Q$, and $Q_1 \subseteq Q$. Based on the same principle, we have $Q_2 \subseteq Q$.

Therefore, solutions of the two submodels in TSM belong to the feasible decision space of ILP. In other words, assume that the solutions of models (2) and (3) are $X_1 = [x_{1opt}^-, ..., x_{kopt}^-, x_{(k+1)opt}^+, ..., x_{nopt}^+]^T$ and $X_2 = [x_{1opt}^+, ..., x_{kopt}^+, x_{(k+1)opt}^-, ..., x_{nopt}^-]^T$, respectively. Then we have $X_1 \in Q$ and $X_2 \in Q$. Let the solutions obtained through TSM be $X_{opt}^{\pm} = [[x_{1opt}^-, x_{1opt}^+], [x_{2opt}^-, x_{2opt}^+], ..., [x_{nopt}^-, x_{nopt}^+]]^T$. Then the question under consideration is whether $X_{opt}^{\pm} \subseteq Q$ or $X_{opt}^{\pm} \not\subseteq Q$.

2.3. Violation Analysis

The example in Huang et al. (1998) has proved that $X_{opt}^{\pm} \subseteq Q$ could be true, which meant that solutions obtained through TSM could be within the feasible decision space. However, we cannot assure that $X_{opt}^{\pm} \subseteq Q$ be always tenable. In some cases, $X_{opt}^{\pm} \not\subseteq Q$ could be true (i.e. $\exists\, X' \in X_{opt}^{\pm}$, such that $X' \notin Q$). This means that some solutions (i.e. X') in $X_{opt}^{\pm}$ do not belong to the feasible decision space of ILP (i.e. *Q*) since they violate the constraint of $A^- X \le B^+$. In this study, such solutions (i.e. X') are denoted as violating solutions. The following numerical example is shown to demonstrate the above issue:

$$\text{Max } f^{\pm} = [3, 3.5]x_1^{\pm} - [1, 1.2]x_2^{\pm}$$
$$(\text{or Min } f^{\pm} = [-3.5, -3]x_1^{\pm} + [1, 1.2]x_2^{\pm}) \tag{8a}$$

Subject to

$$[1, 1.1]x_1^{\pm} + [1.6, 1.8]x_2^{\pm} \le [11.6, 12] \tag{8b}$$

$$[3, 4]x_1^{\pm} - [2, 3]x_2^{\pm} \le [5, 7] \tag{8c}$$

$$x_1^{\pm}, x_2^{\pm} \ge 0 \tag{8d}$$

The following submodels are formulated according to TSM:

$$\text{Max } f^+ = 3.5x_1^+ - x_2^- \tag{9a}$$

Subject to

$$x_1^+ + 1.8x_2^- \le 12 \quad (9b)$$

$$3x_1^+ - 3x_2^- \le 7 \quad (9c)$$

$$x_1^+, x_2^- \ge 0 \quad (9d)$$

Solution of model (9) is $X_1 = [5.79, 3.45]^T$.

$$\text{Max } f^- = 3x_1^- - 1.2x_2^+ \quad (10a)$$

Subject to

$$1.1x_1^- + 1.6x_2^+ \le 11.6 \quad (10b)$$

$$4x_1^- - 2x_2^+ \le 5 \quad (10c)$$

$$0 \le x_1^- \le 5.79 \quad (10d)$$

$$x_2^+ \ge 3.45 \quad (10e)$$

Solution of model (10) is $X_2 = [3.63, 4.76]^T$.

Therefore, solutions obtained through TSM should be $X_{opt}^{\pm} = [[3.63, 5.79], [3.45, 4.76]]^T$. The feasible decision space of model (8) can thus be presented as follows: $Q = \{X \mid [1, 1.6] X \le 12, [3, -3]X \le 7, X \ge 0\}$.

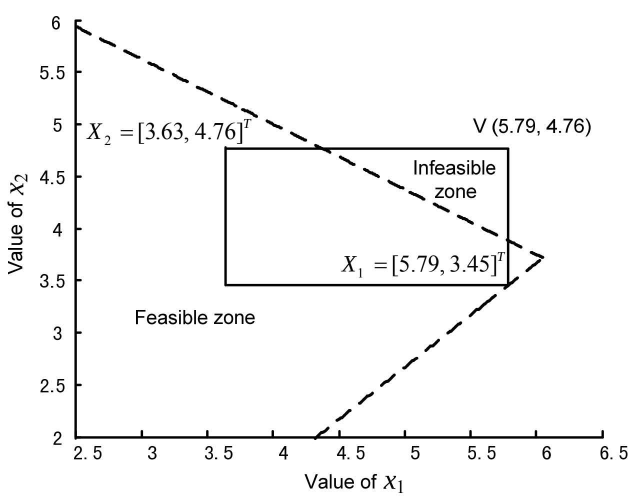

Figure 2. Solutions obtained through TSM.

This is shown in Figure 2, where solutions of models (9) and (10) and the zone of violation are also presented. It is indicated that part of the solutions ($X_{opt}^{\pm}$) are not included within the feasible zone, which means violation of the constraints. Take the point of V = [5.79, 4.76] as an example. We have 1 × 5.79 + 1.6 × 4.76 = 13.406 > 12. Therefore, V is not the feasible solution for ILP model (8) although $V \in X_{opt}^{\pm}$.

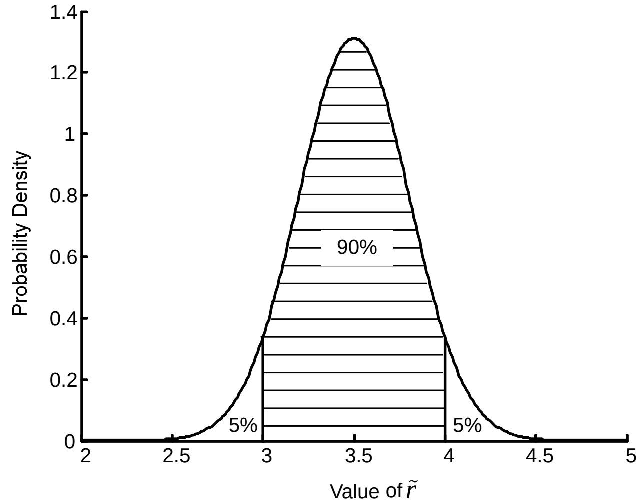

Figure 3. Information of $\tilde{r}$.

2.4. Monte Carcclo Simulation

To further analyze the solutions obtained through TSM and to explore more robust solutions of ILP, Monte Carlo simulation is carried out (Carlin et al., 1992; Tierney and Mira, 1999; Ando et al., 2002; Chib et al., 2002; Fantazzini, 2009). In the data collection process, assume (i) all coefficients are random variables with normal distribution; (ii) the identified interval covers 90% of each random variable. For example, to indentify the daily waste generation rate of a community, a random variable ($\tilde{r}$) which obeys N (3.5, 0.3^2) is acquired according to the historical waste generation records. Then interval [3, 4] which covers 90% of all possible values of $\tilde{r}$ is used as the input for ILP model. Figure 3 shows the obtained distribution information of $\tilde{r}$ and the corresponding interval.

From application point of view, the above assumptions are reasonable, since values near the extremes hold low probability levels. Decision makers can join the process of identifying the representing intervals and determine the coverage rate. It is 90% in the above case; however, such a rate can be adjusted to other levels (e.g. 80 or 95%) according to the practical conditions.

A total of 10,000 Monte Carlo samples for each coefficient are generated and used as inputs for the linear programming model. Figure 4 shows the simulated values. Consistent with the above assumption, about 90% of the generated values lie within the given interval for each coefficient.

The results obtained through the Monte Carlo simulation are presented in Figure 5. It is demonstrated that the definition of feasible decision space is reasonable. The overwhelming majority of the solutions lie in the feasible decision space. Although some solutions are out of the feasible decision space, the probability of their occurrences is low. The existence of constraint violation is related to data collection process. For each coefficient, its extreme values (10%) that are of low occurrences are excluded from consideration if it is presented as an interval. However, these extremes are considered in the Monte Carlo simulation, resulting in the constraint violation.

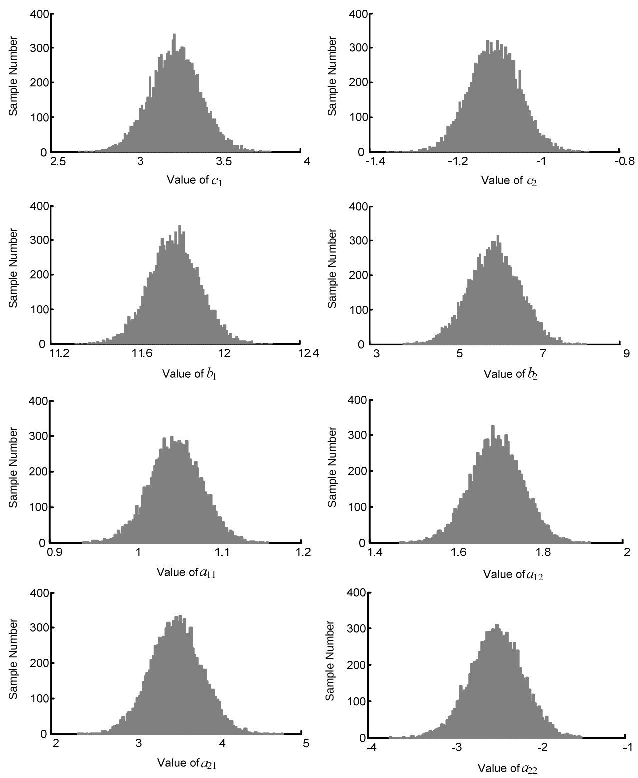

Figure 4. Simulated values for all the coefficients under normal distribution-assumption.

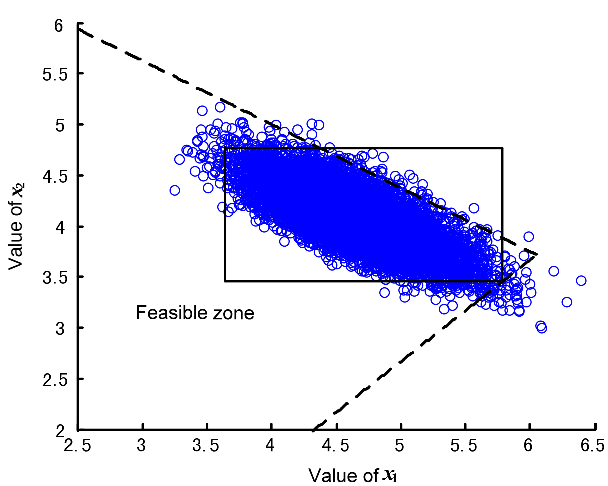

Figure 5. Results of simulation under normal- distribution assumption.

The detailed solutions from TSM also contain several points with violated constraints; however, the relevant probability level is low. Therefore, TSM is generally applicable to problems with their coefficients obeying normal distributions. Moreover, the simulation results indicate that inequalities (5a) to (6b) are tenable. Combinations a_{ij} and x_j used in TSM are based on assumptions (5a) to (6b), which further demonstrates that TSM can be used when all coefficients obey normal distributions.

3. Three-Step Method (ThSM) - an Improvement of TSM

It is demonstrated that TSM, as an interactive algorithm for ILP, aims to generate interval solutions for decision variables. The interval solutions can help decision makers generate a number of possible schemes and to identify desired policies based on further implicit information. The idea of generating interval solutions is one of the key innovations of TSM. However, TSM cannot assure that the obtained solutions toile within the feasible decision space, which may affect its applicability. To deal with this issue, a ThSM approach will be developed. It includes TSM, feasibility test, and constricting method.

3.1. Feasibility Test

In order to identify whether all the solutions obtained through TSM belong to the feasible decision space, the following method of feasibility test is developed. For constraint i (i = 1, 2, …, m), the idea of feasibility test is to find $X^* \in X_{opt}^{\pm}$, such that $\sum_{j=1}^{n} a_{ij}^- x_j^* = \max\{\sum_{j=1}^{n} a_{ij}^- x_j \,|\, x_j \in x_j^{\pm}\}$. Then if $\sum_{j=1}^{n} a_{ij}^- x_j^* \le b_i^+$ is tenable, we have $X_{opt}^{\pm} \subseteq Q$, which means that $X_{opt}^{\pm}$ pass the feasibility test. Otherwise, it means that constraint violation will be generated from $X_{opt}^{\pm}$.

Assume that n_0 ($\le n$) of the n decision variables have interval solutions through TSM. Without loss of generality, assume that solutions for the last n - n_0 decision variables are deterministic numbers (i.e. x_{jopt}), and the first n_0 decision variables hold interval solutions. For the n_0 decision variables, assume that p_i ($p_i \le n_0$) of the corresponding a_{ij}^- are positive, and n_0 - p_i of a_{ij}^- are negative. Here we assume that the first p_i of a_{ij}^- are positive numbers. In other words, we have $a_{ij}^- \ge 0$, j = 1, 2, …, p_i; and $a_{ij}^- < 0$, $j = p_i$ +1, …, n_0. Thus, for $x_j \in x_{jopt}^{\pm}$, we have $a_{ij}^- x_j \le a_{ij}^- x_{jopt}^+$, j = 1, 2,…, p_i, and $a_{ij}^- x_j \le a_{ij}^- x_{jopt}^-$, $j = p_i$ + 1,…, n_0. For $i = 1, 2, ..., m$, $A^- X = a_{i1}^- x_1 + ... + a_{in}^- x_n \le \sum_{j=1}^{p_i} a_{ij}^- x_{jopt}^+ + \sum_{j=p_i+1}^{n_0} a_{ij}^- x_{jopt}^- + \sum_{j=n_0+1}^{n} a_{ij}^- x_{jopt}$.

For constraint i, we have $X^* = [x_{1opt}^+, ..., x_{p_i opt}^+, x_{(p_i+1)opt}^-, ..., x_{n_0 opt}^-, x_{(n_0+1)opt}, ..., x_{nopt}]^T$. Therefore, if $(A^-)_i X^* \le b_i^+$, i = 1, 2, …, m, we have $X_{opt}^{\pm} \in Q$ and $X_{opt}^{\pm}$ pass the feasibility test. Otherwise, if $(A^-)_i X^* > b_i^+$, then infeasible solutions will be included in $X_{opt}^{\pm}$. At least X^* will violate the constraints. Therefore, solutions obtained through TSM ($X_{opt}^{\pm}$) need to be revised.

3.2. Constricting Method

3.2.1. Definitions

For solutions which do not pass the feasibility test, the following constricting method should be adopted: Let M($X_{opt}^{\pm}$) $= [0.5(x_{1opt}^- + x_{1opt}^+), ..., 0.5(x_{nopt}^- + x_{nopt}^+)]^T$, D($X_{opt}^{\pm}$) = $[0.5(x_{1opt}^+ - x_{1opt}^-), ..., 0.5(x_{nopt}^+ - x_{nopt}^-)]^T$, and $\vec{Q} = [q_1, q_2, …, q_n]^T$ where $0 \le q_j \le 1$, $q_j \in R$, j = 1, 2, …, n.

For the convenience of expressing formulas in the following parts, we further define the following:

M = $[m_1, m_2, \ldots, m_n]^T$ = M($X_{opt}^{\pm}$), where $m_j = 0.5(x_{jopt}^- + x_{jopt}^+)$, $j = 1, 2, \ldots, n$; (11a)

D = $[d_1, d_2, \ldots, d_n]^T$ = D($X_{opt}^{\pm}$), where $d_j = 0.5(x_{jopt}^+ - x_{jopt}^-)$, $j = 1, 2, \ldots, n$; (11b)

$Y^{\pm} = (y_j^{\pm})_{n \times 1}$, where $y_j^{\pm} = [m_j - q_j d_j, m_j + q_j d_j]$, $j = 1, 2, \ldots, n$. (12)

Remark 2. M($X_{opt}^{\pm}$) $\in Q$ (M $\in Q$).

M($X_{opt}^{\pm}$) ≥ 0 is straightforward For $i = 1, 2, \ldots, m$, we have $a_{i1}^-(0.5x_{1opt}^- + 0.5x_{1opt}^+) + \ldots + a_{in}^-(0.5x_{nopt}^- + 0.5x_{nopt}^+) = 0.5\sum_{j=1}^{k} a_{ij}^- x_{jopt}^- + 0.5\sum_{j=1}^{k} a_{ij}^- x_{jopt}^+ + 0.5\sum_{j=k+1}^{n} a_{ij}^- x_{jopt}^- + 0.5\sum_{j=k+1}^{n} a_{ij}^- x_{jopt}^+ \leq 0.5[\sum_{j=1}^{k} |a_{ij}^{\pm}|^+ Sign(a_{ij}^{\pm}) x_{jopt}^- + \sum_{j=k+1}^{n} |a_{ij}^{\pm}|^- Sign(a_{ij}^{\pm}) x_{jopt}^+] \leq 0.5b_i^+ + 0.5b_i^- \leq b_i^+$ Therefore, A^- M($X_{opt}^{\pm}$) $\leq B^+$, and thus M($X_{opt}^{\pm}$) $\in Q$ is tenable.

Obviously, m_j is the center of the obtained $x_{jopt}^{\pm}$ for the j^{th} decision variable, and d_j is named as radius since it equals half-width of $x_{jopt}^{\pm}$. In other words, d_j shows the distance between the endpoint and the center of $x_{jopt}^{\pm}$. Accordingly, M and D respectively present the center and radius of $X_{opt}^{\pm}$. As defined in formula (12), $Y^{\pm}$ stands for the constricted interval of $X_{opt}^{\pm}$. Obviously, when $Q = [1, \ldots, 1]^T$, then $Y^{\pm} = X_{opt}^{\pm}$; when $Q = [0, \ldots, 0]^T$, then $Y^{\pm}$ = M. According to Remark 2, we then know M $\in$ Q. The larger the value of Q, the wider of $Y^{\pm}$; this results higher possibility for $Y^{\pm}$ to contain infeasible solutions. Therefore, the objective of the constricting method is to identify the value of Q, so that the corresponding $Y^{\pm}$ could lie within the feasible decision space. In detail, $y_j^{\pm}$ is the shrunk interval of $x_{jopt}^{\pm}$, and the constricting rate depends on the value of q_j. When $q_j = 1$, it indicates that $y_j^{\pm} = x_{jopt}^{\pm}$, which means that $x_{jopt}^{\pm}$ is not constricted at all. When $q_j = 0$, it shows that $y_j^{\pm} = m_j$, which means that interval $x_{jopt}^{\pm}$ is constricted into a deterministic number, i.e. its central value.

However, solutions of some decision variables obtained through TSM are deterministic numbers. In other words, these solutions do not hold any capacity/space to be constricted. Therefore, the constricting method cannot be applied to such variables. Without loss of generality, assume that solutions for the last $n - n_0$ variables are deterministic numbers (i.e. m_j), and the first n_0 decision variables are intervals. This means that $q_j = 0$ and $y_j^{\pm} = m_j$ when $j = n_0 + 1, n_0 + 2, \ldots, n$. Thus, the problem under consideration is to generate appropriate values for q_j, where $j = 1, 2, \ldots, n_0$. Then the values for $Y^{\pm}$ can be calculated according to formula (12).

3.2.2. Constraints

In order to indentify the values for q_j ($j = 1, 2, \ldots, n_0$), a new programming model will be formulated. Constraints for q_j depend on constraints of $Y^{\pm}$, since $Y^{\pm}$ should lie within the feasible decision space of the ILP problem. This means that the following inequality should be held:

$$A^- Y \leq B^+, \forall Y \in Y^{\pm} \quad (13)$$

In detail, we have:

$$a_{i1}^- y_1 + a_{i2}^- y_2 + \ldots + a_{in}^- y_n \leq b_i^+ \quad (14)$$

where $y_j \in y_j^{\pm}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

As mentioned in section 3.1, we need to find $Y^* \in Y^{\pm}$, such that $\sum_{j=1}^{n} a_{ij}^- y_j^* = \max\{\sum_{j=1}^{n} a_{ij}^- y_j \,|\, y_j \in y_j^{\pm}\}$. Constraints (13) and (14) will be tenable only if $\sum_{j=1}^{n} a_{ij}^- y_j^* \leq b_i^+$ is tenable. Assume that the first p_i of a_{ij}^- are positive numbers and the remaining $n_0 - p_i$ of a_{ij}^- are negative ones in the i^{th} constraint. Then we have: $\sum_{j=1}^{p_i} a_{ij}^- y_j \leq \sum_{j=1}^{p_i} a_{ij}^- y_j^+$, and $\sum_{j=p_i+1}^{n_0} a_{ij}^- y_j \leq \sum_{j=p_i+1}^{n_0} a_{ij}^- y_j^-$. Therefore, $a_{i1}^- y_1 + \ldots + a_{in}^- y_n \leq \sum_{j=1}^{p_i} a_{ij}^- y_j^+ + \sum_{j=p_i+1}^{n_0} a_{ij}^- y_j^- + \sum_{j=n_0+1}^{n} a_{ij}^- m_j$.

For constraint i, we have $Y^* = [y_{1opt}^+, \ldots, y_{p_i opt}^+, y_{(p_i+1)opt}^-, \ldots, y_{n_0 opt}^-, m_{(n_0+1)opt}, \ldots, m_{nopt}]^T$. Let $(A^-)_i Y^* \leq b_i^+$, we have:

$$\sum_{j=1}^{p_i} a_{ij}^-(m_j + q_j d_j) + \sum_{j=p_i+1}^{n_0} a_{ij}^-(m_j - q_j d_j) + \sum_{j=n_0+1}^{n} a_{ij}^- m_j \leq b_i^+ \quad (15)$$

To express constraints (15) briefly, we have:

$$\sum_{j=1}^{p_i} a_{ij}^- q_j d_j - \sum_{j=p_i+1}^{n_0} a_{ij}^- q_j d_j \leq b_i^+ - \sum_{j=1}^{n} a_{ij}^- m_j \quad (16)$$

Thus the constraints for solving q_j ($j = 1, 2, \ldots, n_0$) have been formulated. In the next step, the objective function will be established.

3.2.3. Objectives

The constricting method is based on the solutions obtained through TSM. Since the interval solution for each decision variable will be constricted, the corresponding solution for the objective function value (1a) will be constricted as well. Therefore, the obtained value of objective (1a) in the constricting method will be included within the interval of TSM solution.

Assume the value of objective (1a) obtained through the constricting method is [f_1^-, f_1^+]. Then we have $f_{opt}^- \leq f_1^- \leq f_1^+ \leq f_{opt}^+$, where[f_{opt}^-, f_{opt}^+] is the solution obtained through TSM. Thus, it is not meaningful to use the original objective in the constricting method. Moreover, if the objective of model (2) is used in the constricting method, we then have:

$$f^- = \sum_{j=1}^{k} c_j^- x_j^- + \sum_{j=k+1}^{n} c_j^- x_j^+ = \sum_{i=1}^{n} c_j^- m_j - \sum_{i=1}^{k} c_j^- d_j q_j + \sum_{i=k+1}^{n} c_j^- d_j q_j \quad (17)$$

To minimize f^- , we will get $q_j = 0$ where $j = k + 1, k + 2, \ldots, n$. This means that the values of q_j for the decision variables with negative coefficients will be zero, and thus such decision variable will be deterministic numbers. Similarly, if the objective of model (3) is used, we will have:

$$f^+ = \sum_{j=1}^{k} c_j^- x_j^+ + \sum_{j=k+1}^{n} c_j^- x_j^- = \sum_{i=1}^{n} c_j^- m_j + \sum_{i=1}^{k} c_j^- d_j q_j - \sum_{i=k+1}^{n} c_j^- d_j q_j \quad (18)$$

To minimize f^+, we will get $q_j = 0$ where j = 1, 2, …, k. In this case, the values of q_j for decision variables with positive coefficients will be zero, and thus such decision variable will be deterministic numbers.

If objective (2a) or (3a) is used in the constricting method, the number of interval solutions for decision variables will be reduced. Due to the above reasons, a new objective function should be developed in the constricting method.

In fact, the smaller the value of q_j, the less possible for $Y^\pm$ [$Y^\pm = (y_j^\pm)_{n\times1}$] contain infeasible solutions. On the other hand, considering the convenience of decision makers, the larger the width of $Y^\pm$ is, the more flexible of the generated schemes are. Thus, a large value of q_j is preferred. Figure 6 shows the obtained $Y^\pm$ under different values of q_j. The assumption of $q_1 = q_2 = q$ is used in the example. Since constraint (16) have assured that the obtained $Y^\pm$ belongs to a feasible zone the objective is then to maximize the value of Q.

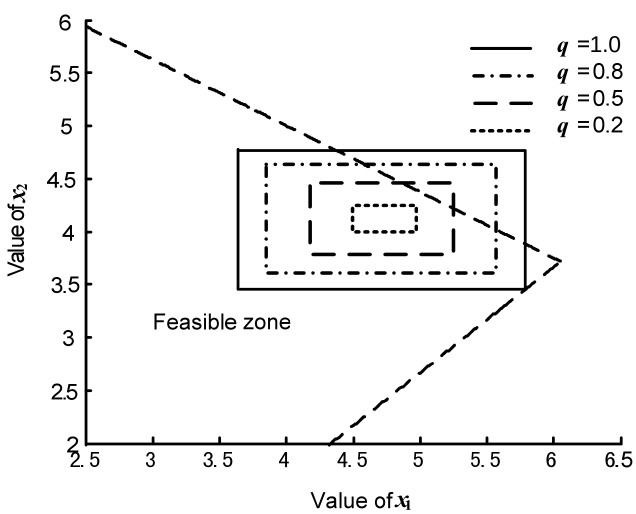

Figure 6. Solutions under different q-levels.

Two types of objective expressions are considered. One is to assume that $q_i = q_j = q$, where i, j =1, 2, …, n_0, and $i \neq j$. The other is to maximize $2q_1d_1 \times 2q_2d_2 \times \ldots \times 2q_{n_0}d_{n_0}$ which can be replaced by $q_1 \times q_2 \times \ldots \times q_{n_0}$. For the linear programming problem with two decision variables having interval solutions, the objective of $2q_1d_1 \times 2q_2d_2$ [$(y_1^+ - y_1^-) \times (y_2^+ - y_2^-)$] means to maximize the area of the obtained solution. Thus, the linear programming models for solving q_j (j =1, 2, …, n_0) can be presented as follows:

$$\text{Max } q \tag{19a}$$

Subject to

$$\sum_{j=1}^{p_i} a_{ij}^- q d_j - \sum_{j=p_i+1}^{n_0} a_{ij}^- q d_j \le b_i^+ - \sum_{j=1}^{n} a_{ij}^- m_j \tag{19b}$$

$$0 \le q \le 1 \tag{19c}$$

where i = 1, 2,…, m; j = 1, 2, …, n_0.

$$\text{Max } q_1 \times q_2 \times \ldots \times q_{n_0} \tag{20a}$$

Subject to

$$\sum_{j=1}^{p_i} a_{ij}^- q_j d_j - \sum_{j=p_i+1}^{n_0} a_{ij}^- q_j d_j \le b_i^+ - \sum_{j=1}^{n} a_{ij}^- m_j \tag{20b}$$

$$0 \le q_j \le 1 \tag{20c}$$

where i = 1, 2,…, m and j = 1, 2, …, n_0.

Solutions of q_{jopt} (j = 1, 2, …, n) can be obtained through solution of model (20). Then according to formula (12), we have $Y_{opt}^\pm$ = ($y_{jopt}^\pm$)$_{n\times1}$. The solution method with the objective being expressed as (19a) is named ThSM-I, while the one with objective (20a) is named ThSM-II.

3.3. Numerical Example

A simplified example is introduced to show the solution processes of ThSM-I and ThSM-II in detail:

$$\text{Max } f^\pm = [2, 2.4]x_1^\pm - [1, 1.3]x_2^\pm + [1.5, 1.8]x_3^\pm \tag{21a}$$
$$(\text{Min } f^\pm = [-2.4, -2]x_1^\pm + [1, 1.3]x_2^\pm + [-1.8, -1.5]x_3^\pm)$$

Subject to

$$[2.6, 3.5]x_1^\pm + [2, 2.4]x_2^\pm + [3.2, 3.8]x_3^\pm \le [18, 22] \tag{21b}$$

$$[4.6, 5.5]x_1^\pm + [3, 3.6]x_2^\pm - [1.3, 1.6]x_3^\pm \le [8, 9] \tag{21c}$$

$$[1, 1.3]x_1^\pm - [6, 6.5]x_2^\pm + [2, 2.5]x_3^\pm \le [2.2, 2.6] \tag{21d}$$

$$x_1^\pm, x_2^\pm, x_3^\pm \ge 0 \tag{21e}$$

Step 1. Two-step method (TSM)

Through TSM, we can obtain the following solutions: $x_{1opt}^\pm$ = [1.56, 2.18], $x_{2opt}^\pm$ = 1.22, and $x_{3opt}^\pm$ = [2.66, 4.18]. The corresponding objective function value is $f_{opt}^\pm$ = [5.51, 11.55].

Step 2. Feasibility Test

The following inequalities should be tested:

$$2.6x_{1opt}^+ + 2x_{2opt}^+ + 3.2x_{3opt}^+ \le 22 \tag{22a}$$

$$4.6x_{1opt}^+ + 3x_{2opt}^+ - 1.6x_{3opt}^- \le 9 \tag{22b}$$

$$1x_{1opt}^+ - 6.5x_{2opt}^- + 2x_{3opt}^+ \le 2.6 \tag{22c}$$

Take the results obtained through the TSM as inputs of inequalities (22a) to (22c), we find inequality (22b) is not tenable. Thus, the solutions of TSM do not pass the test. The constricting method should then be used to revise the results.

Step 3. Constricting Method

According to the solutions of TSM, we have: M = M($X_{opt}^{\pm}$) = [1.87, 1.22, 3.42]^T, and D = D($X_{opt}^{\pm}$) = [0.31, 0, 0.76]^T. Assume that $Y^{\pm} = [y_1^{\pm}, y_2^{\pm}, y_3^{\pm}]^T$ is the new solution, where y_1 = [1.87 - 0.31q_1, 1.87 + 0.31q_1], y_2 = 1.22 (q_2 = 0), and y_3 = [3.42 - 0.76q_3, 3.42 + 0.76q_3]. Then, according to ThSM-I [model (19)], we have:

$$\text{Max } q \tag{23a}$$

Subject to

$$2.6 \times 0.31q + 3.2 \times 0.76q \le 22 - (2.6 \times 1.87 + 2 \times 1.22 + 3.2 \times 3.42) \tag{23b}$$

$$4.6 \times 0.31q - (-1.6) \times 0.76q \le 9 - (4.6 \times 1.87 + 3 \times 1.22 - 1.6 \times 3.42) \tag{23c}$$

$$1 \times 0.31q + 2 \times 0.76q \le 2.6 - (1 \times 1.87 - 6.5 \times 1.22 + 2 \times 3.42) \tag{23d}$$

$$0 \le q \le 1 \tag{23e}$$

Solutions of model (23) are q = 0.84. Thus we can get the revised solutions for model (21): $y_{1opt}^{\pm} = [1.61, 2.13]$, $y_{2opt}^{\pm}$ = 1.22, and $y_{3opt}^{\pm}$ = [2.78, 4.06]. The corresponding objective function value is $f_{opt}^{\pm}$ = [5.804, 11.2]. According to ThSM-II [model (20)], we have:

$$\text{Max } q_1 q_3 \tag{24a}$$

Subject to

$$2.6 \times 0.31q_1 + 3.2 \times 0.76q_3 \le 22 - (2.6 \times 1.87 + 2 \times 1.22 + 3.2 \times 3.42) \tag{24b}$$

$$4.6 \times 0.31q_1 - (-1.6) \times 0.76q_3 \le 9 - (4.6 \times 1.87 + 3 \times 1.22 - 1.6 \times 3.42) \tag{24c}$$

$$1 \times 0.31q_1 + 2 \times 0.76q_3 \le 2.6 - (1 \times 1.87 - 6.5 \times 1.22 + 2 \times 3.42) \tag{24d}$$

$$0 \le q_1, q_3 \le 1 \tag{24e}$$

Solutions of model (24) are q_1 = 0.77, q_3 = 0.91. Thus we have $y_{1opt}^{\pm}$ = [1.63, 2.11], $y_{2opt}^{\pm}$ = 1.22, and $y_{3opt}^{\pm}$ = [2.73, 4.11]. The corresponding objective function value is $f_{opt}^{\pm}$ = [5.769, 11.242].

Interval solution for the objective function obtained by ThSM-II (i.e. [5.769, 11.242]) holds a larger width, compared with that of ThSM-I (i.e. [5.804, 11.2]). Moreover, the constricting ratio for each decision variable can be adjusted independently considering its contribution to the objective in ThSM-II. Therefore, a larger value of f^+ can be obtained, although ThSM-II results in a relatively smaller f^- than ThSM-I. Since the objective is to maximize the objective function, f^+ is considered more important than f^-. Thus, the algorithm of ThSM-II is preferred. However, it is assumed that decision variables with interval solutions hold the same constricting ratio in ThSM-I; therefore, the formulated model for q_j is linear. Thus, the advantage of ThSM-I is its low computational requirement.

Another important feature of ThSM-I and ThSM-II is that their solutions are included within those of TSM. In detail, the value for each decision variable obtained through TSM covers those of ThSM-I and ThSM-II; the objective function value of TSM covers those of ThSM-I and ThSM-II. If decision makers prefer obtaining interval solutions with large widths, TSM can be used. However, a potential system-failure risk may exist since infeasible solutions may be included. All schemes generated from solutions of ThSM-I and ThSM-II are feasible, although the obtained interval for each decision variable is narrower than that of TSM. In real-world cases, decision makers can select appropriate solution methods according to their preferences and practical conditions.

4. Discussions

The results of Monte Carlo simulation with the assumption of normal distribution demonstrate that solution of TSM is applicable to real-world cases where uncertain input coefficients obey normal distribution. However, no distribution information of the parameters is available in ILP. In other words, an arbitrary type of distribution function could be possible for the coefficients. To explore the solutions of ILP, model (8) is further examined. The simulation is based on two assumptions. Firstly, it is assumed that all of the coefficients in the ILP model are random variables, although their distribution functions are unknown. Secondly, it is assumed that the random variables take values from the given intervals. For example, when a parameter is assumed to obey a uniform distribution, random variable c_1 could take any value in [3.0, 3.5] with the same probability.

A total of 10,000 samples for each coefficient are generated and used as inputs for model (8). Three scenarios corresponding with three assumed distribution functions are introduced, besides the normal distribution-assumption. In scenario 1, all input coefficients obey uniform distribution; in scenarios 2 and 3, all input coefficients obey chi-square distribution and con-chi-square distribution, respectively. Take a_{21} as an example. Figure 7 shows the three types of simulated samples for a_{21} corresponding with three distribution functions. It indicates that the considered distribution functions are typical. In scenario 1, all values within the interval hold similar probability levels; in scenario 2, the values approaching the left endpoint

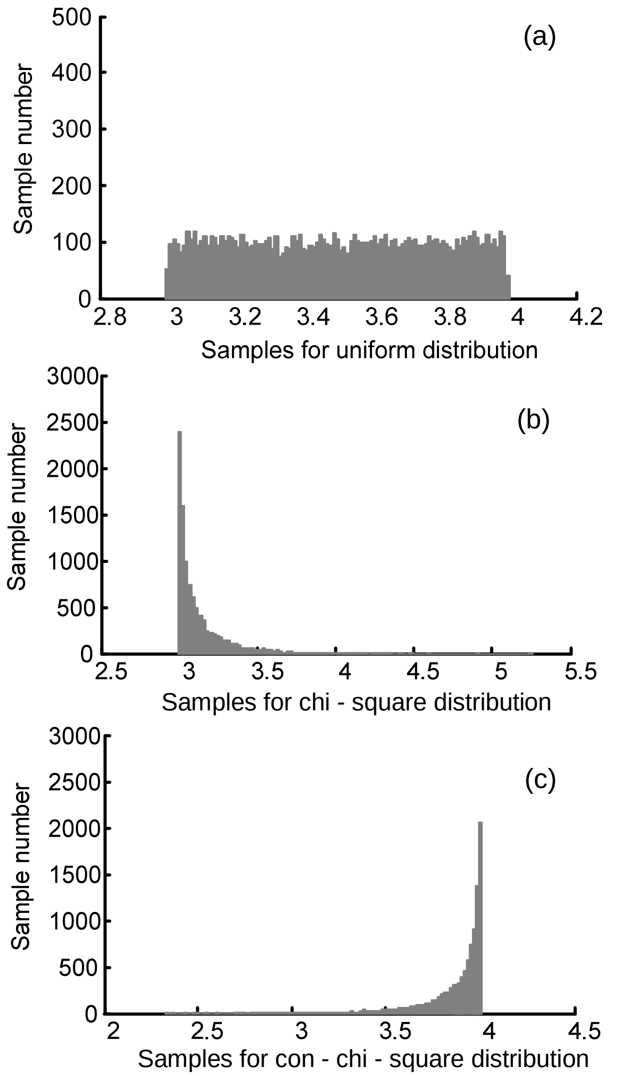

Figure 7. Samples under different distribution assumptions: (a) samples under Scenario 1; (b) samples under Scenario 2; (c) samples under Scenario 3.

hold higher probabilities. Meanwhile, the values approaching the right endpoint hold higher probabilities in scenario 3. Figure 8 presents the results of Monte Carlo simulations. For the convenience of comparison, the TSM solutions are also presented in Figure 8.

The simulation result of scenario 1 is similar to that under normal distribution assumption as shown in Figure 5. The main difference lies in that no infeasible solution exists under the uniform-distribution assumption as shown in Figure 8a. This characteristic is due to the feature of uniform distribution. As shown in Figure 7a, only values within the given interval can be taken under the assumption of uniform distribution; thereofre, no infeasible solution is allowed in the Monte Carlo simulation. Figures 8b and 8c show the simulation results under assumptions of chi-square and con-chi-square distributions. It indicates that infeasible solutions exist under these assumptions. According to Figures 7b and 7c, most of the solution values are within

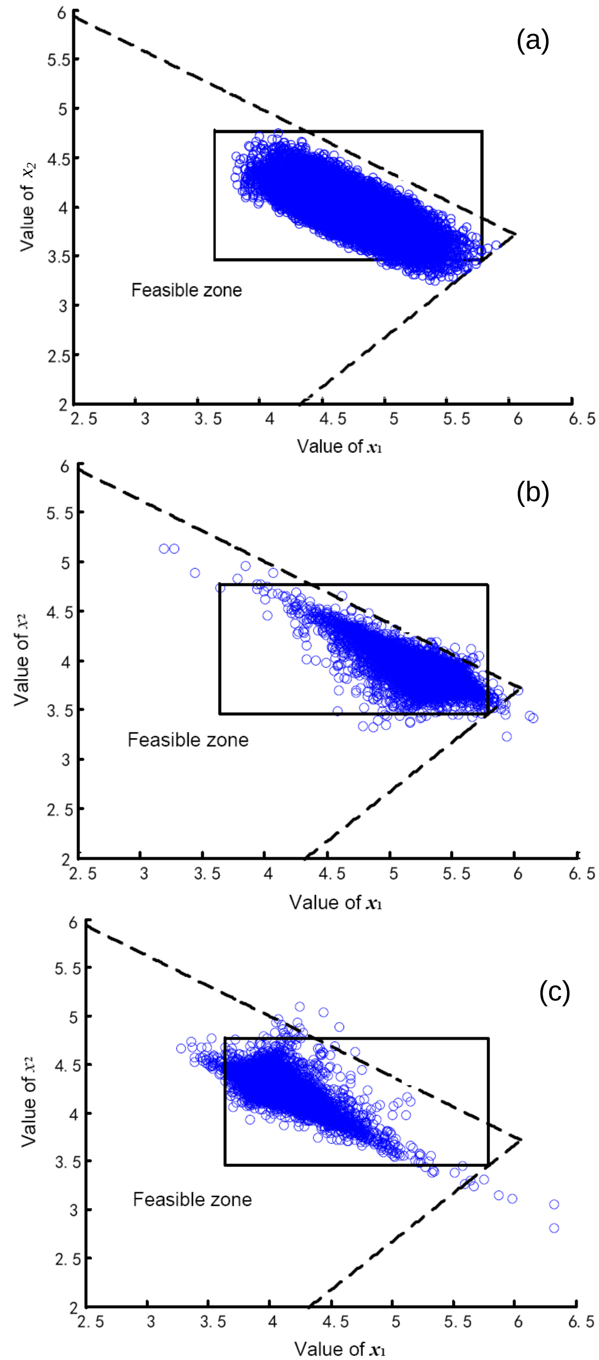

Figure 8. Results of simulation under different distribution assumptions: (a) simulation results under scenario 1; (b) simulation results under scenario 2; (c) simulation results under scenario 3.

the given interval; however, there is a minor probability for the values to be out of the interval in the Monte Carlo simulation. Thus, some infeasible solutions exist under scenarios 2 and 3.

Comparing the results under scenarios 2 and 3 as shown in Figures 8b and 8c, solutions of ILP could be significantly different when the distribution functions of coefficients are different. The uncertainties in the solutions come from the uncertain input parameters. In the numerical example, it is difficult to indentify interval solutions for x_1 and x_2 that satisfy both chi-square and con-chi-square distributions for the parameters. However, parameters in ILP could obey any type of distribution. For a general ILP problem with n decision variables, we can hardly indentify an interval solution for each decision variable that is feasible under an arbitrary distribution assumption for the coefficients. Therefore, assumptions of distribution types for the coefficients are necessary and should be presented when the solutions of ILP are analyzed.

According to the above analysis, it is reasonable to present the distribution types when the results of TSM are analyzed. As mentioned in Section 2, when all the coefficients obey normal distributions, inequalities (5a) to (6b) are tenable and TSM can be used to solve such an ILP problem. In scenario 1, all of the coefficients obey uniform distributions; the simulation results show that $a_{ij} x_j$ holds the characteristics of inequalities (5a) to (6b). Thus TSM can be used under scenario 1. As a result, ThSM-I and ThSM-II can be used to solve such ILP problems. However, under scenarios 2 and 3, the simulation results demonstrate that inequalities (5a) to (6b) are not tenable. It means that the assumptions of holding combinations of a_{ij} and x_j as used in TSM cannot be satisfied. Thus TSM cannot be used when the coefficients of ILP obey chi-square or con-chi-square distribution. Consequently, ThSM-I and ThSM-II cannot be used since they are based on the solutions of TSM.

In general, the developed ThSM-I and ThSM-II are applicable when coefficients of ILP obey normal or uniform distribution. Under other scenarios, such as chi-square and con-chi-square distributions, ThSM-I and ThSM-II cannot be used. To solve the problem, one way is to develop new solution methods with assumptions of other distributions. For example, a new solution method could be developed to deal with problems with their coefficients obeying chi-square distributions. In real-world cases, coefficients of an ILP model could hold different distribution functions (e.g. a_{11} obeys normal distribution, and a_{12} obeys chi-square distribution). We may then indentify an "equivalent normal distribution" for a_{12}. Then ThSM-I and ThSM-II can be adopted to solve the problem. However, many questions may still exist, such as: how to transform a chi-square distribution to a normal one? What is the definition of "equivalent normal distribution"? What are the principals of such transformations? Further studies could focuse on these issues.

5. Conclusions

The two-step method (TSM) which was used widely to solve the interval linear programming (ILP) models has been analyzed. The principals and assumptions used in TSM are presented and discussed. The definition of feasible decision space for ILP has been given. Also existence of infeasible solutions and how these solutions are generated have been examined.

To analyze whether results obtained through TSM contain infeasible solution, feasibility test is needed. A constricting method has been developed to eliminate the infeasible part of the solution through constricting the solutions obtained through TSM. Based on three proposed steps (TSM, feasibility test, and constricting method), two new solution methods for ILP [named three-step method-I (ThSM-I), and three-step method-II (ThSM-II)] have been developed. The main advantage of ThSM-I and ThSM-II is that no infeasible solutions would be included in the obtained results. Moreover, the developed methods could generated interval solutions and do not have high computational requirements. An example has been presented to explain in detail the solution processes of ThSM-I and ThSM-II.

In addition, three scenarios of Monte Carlo simulations have been introduced to explore the detailed solutions for ILP. It indicates that the developed methods are applicable when all the coefficients of ILP are assumed to obey normal or uniform distribution. Under other assumptions of distribution functions (e.g. chi-square distribution), ThSM-I and ThSM-II cannot be used. Further studies should be developed to deal with these cases.

Acknowledgments. This research was supported by the Major Science and Technology Program for Water Pollution Control and Treatment (2009ZX07104-004).

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